M334 PART VI

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THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART VI GEOMETRY OF SURFACES IN E^3

DIFFERENTIAL GEOMETRY

LANCASHIRE POLYTECHNIC LLRS

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART VI

GEOMETRY OF SURFACES IN E^3

A commentary on Chapter VI of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition, (W.A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero, Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

VI.1 INTRODUCTION AND THE FUNDAMENTAL EQUATIONS

Introduction

In this chapter we draw together all the results that we have developed in this course. In this section we use frame fields, introduced in Section II.6, their structure equations from Section II.8, the calculus of forms on surfaces from Section IV.4, orientability from Section IV.7 and the theory of the shape operator from Chapter V. You should be familiar with these topics before tackling this section.

In Chapters I and IV we developed two different approaches to differential forms. In Chapter I we introduced forms in E^3 in terms of simple forms such as dx and dx dy, while in Chapter IV we introduced differential forms on a surface in E^3, $M \subset E^3$, in a coordinate independent manner. We showed that the two approaches gave structures with similar properties. In this chapter we want to be able to use both approaches interchangeably. We want to use the structural equations, that were obtained in terms of coordinates in Section II.8, by interpreting the forms as linear functionals and bilinear forms operating on tangent vector fields. For now, accept that the two approaches are equivalent. In the Optional Text we indicate why this is so and give a more abstract approach to some of the results of this section. The Optional Text is not examinable so do not waste any time on it if you find it difficult.

READ: The Introduction to Chapter VI and Section VI.1 (pages 245-250).

Optional Text

It is possible to show that the approach to forms developed in Chapter I is equivalent to that developed in Chapter IV and consequently that we can interpret the forms in Theorem VI.1.7 as linear functionals and bilinear forms. It will be necessary to interpret them this way in subsequent sections.

Rather than prove that the two approaches are equivalent we shall indicate how they both follow from a more abstract approach. The approach of Chapter I is in terms of coordinates and the approach of Chapter IV uses patches; a more abstract approach uses the *Poisson bracket* of vector fields.

For our purposes the most useful definition is given in Exercise 7, page 81. For vector fields V, W the Poisson bracket is the vector field

$$[V, W] = \nabla_V W - \nabla_W V.$$

Part (a) of Exercise 7 tells us that

$$[V, W] = VW - WV,$$

from which we deduce that, given a vector field $Z = (z_1, z_2, z_3)$,

$$\begin{aligned}
\nabla_{[V, W]} Z &= \sum_{i=1}^{3} ([V, W][z_i])U_i \\
&= \sum_{i=1}^{3} ((VW - WV)[z_i])U_i \\
&= \Sigma V(W[z_i])U_i - \Sigma W(V[z_i])U_i \\
&= \nabla_V(\Sigma W[z_i]U_i) - \nabla_W(\Sigma V[z_i]U_i) \\
&= \nabla_V(\nabla_W Z) - \nabla_W(\nabla_V Z) \\
&= (\nabla_V \nabla_W - \nabla_W \nabla_V)(Z),
\end{aligned}$$

where $\nabla_V\nabla_W$ and $\nabla_W\nabla_V$ represent the appropriate composite mappings.

Hence

$$\nabla_{[V,W]} = \nabla_V\nabla_W - \nabla_W\nabla_V.$$

When we are dealing with a surface M in E^3 we need to know that the Poisson bracket of two tangent vector fields is again a tangent vector field. This is proved in Exercise 9, page 195.

Now we are in a position to give an abstract definition of the differential $d\phi$ of a 1-form ϕ (on a surface). We evaluate the form on a pair of (tangent) vector fields as follows:

$$d\phi(V, W) = V[\phi(W)] - W[\phi(V)] - \phi([V, W]),$$

where here we are using directional derivatives.

Now we can deal with the equations of Theorem VI.1.7 without using Theorem II.8.3. For instance the first equation of part (1) is proved by applying both sides of the equation to the vector fields E_1 and E_2. For example,

$$\begin{aligned} d\theta_1(E_1, E_2) &= E_1[\theta_1(E_2)] - E_2[\theta_1(E_1)] - \theta_1([E_1, E_2]) \\ &= E_1[0] - E_2[1] - \theta_1([E_1, E_2]) \\ &= 0 - \theta_1(\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \\ &= -E_1\cdot\nabla_{E_1}E_2 + E_1\cdot\nabla_{E_2}E_1. \end{aligned}$$

Since $E_1\cdot E_2 = 0$ and $E_1\cdot E_1 = 1$ it follows that

$$E_1[E_1\cdot E_2] = 0 = E_2[E_1\cdot E_1],$$

and from the Leibnizian property of the covariant derivative (Corollary II.5.4, page 80) we can deduce that

$$\nabla_{E_1}E_1\cdot E_2 + E_1\cdot\nabla_{E_1}E_2 = 0$$

and

$$\nabla_{E_2}E_1\cdot E_1 + E_1\cdot\nabla_{E_2}E_1 = 2E_1\cdot\nabla_{E_2}E_1 = 0.$$

Hence

$$\begin{aligned} d\theta_1(E_1, E_2) &= E_2\cdot\nabla_{E_1}E_1 \\ &= \omega_{12}(E_1), \qquad \text{by Lemma II.7.1.} \end{aligned}$$

Also

$$\begin{aligned} (\omega_{12}\wedge\theta_2)(E_1, E_2) &= \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1) \\ &= \omega_{12}(E_1). \end{aligned}$$

Equations like part (3) are proved by applying both sides of the equation to the pair of tangent vector fields $\{E_1, E_2\}$. We find

$$\begin{aligned} d\omega_{12}(E_1, E_2) &= E_1[\omega_{12}(E_2)] - E_2[\omega_{12}(E_1)] - \omega_{12}([E_1, E_2]) \\ &= E_1[E_2\cdot\nabla_{E_2}E_1] - E_2[E_2\cdot\nabla_{E_1}E_1] - E_2\cdot\nabla_{[E_1, E_2]}E_1 \\ &= \nabla_{E_1}E_2\cdot\nabla_{E_2}E_1 + E_2\cdot\nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}E_2\cdot\nabla_{E_1}E_1 \\ &\quad - E_2\cdot\nabla_{E_2}\nabla_{E_1}E_1 - E_2\cdot(\nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}\nabla_{E_1}E_1) \end{aligned}$$

$$= \nabla_{E_1} E_2 \cdot \nabla_{E_2} E_1 - \nabla_{E_2} E_2 \cdot \nabla_{E_1} E_1$$

$$= (\omega_{21}(E_1)E_1 + \omega_{23}(E_1)E_3) \cdot (\omega_{12}(E_2)E_2 + \omega_{13}(E_2)E_3)$$

$$- (\omega_{21}(E_2)E_1 + \omega_{23}(E_2)E_3) \cdot (\omega_{12}(E_1)E_2 + \omega_{13}(E_1)E_3)$$

$$= \omega_{23}(E_1)\omega_{13}(E_2) - \omega_{23}(E_2)\omega_{13}(E_1)$$

$$= (\omega_{13} \wedge \omega_{32})(E_1, E_2).$$

Now you can see why O'Neill has avoided such proofs, although they can be simplified by careful use of suffices.

It remains to show only that this new approach to the differential of a 1-form includes those of both Chapters I and IV.

How does this give us the formulas of Chapter I? Firstly, in E^3 we notice that, since the standard frame fields are parallel, the Poisson brackets $[U_i, U_j]$ are zero. Hence if $\phi = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ then

$$d\phi(U_1, U_2) = U_1[\phi(U_2)] - U_2[\phi(U_1)]$$

$$= U_1[a_2] - U_2[a_1]$$

$$= \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}.$$

But, using a definition of the wedge product similar to that used in Chapter IV,

$$dx_1 dx_2 (U_1, U_2) = dx_1(U_1)dx_2(U_2) - dx_1(U_2)dx_2(U_1)$$

$$= 1.1 - 0.0 = 1.$$

Using these and similar results we see that

$$d\phi = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 dx_2 + \text{two similar terms.}$$

How do we obtain the formula of Definition IV.4.4? Firstly, if we are dealing with a patch x we note that the Poisson bracket $[x_u \circ x^{-1}, x_v \circ x^{-1}]$ of the partial velocity tangent vector fields is zero. This follows since at the point $x(u_0, v_0)$

$$[x_u \circ x^{-1}, x_v \circ x^{-1}]\,(x(u_0, v_0))$$

$$= \nabla_{x_u \circ x^{-1}(x(u_0, v_0))}(x_v \circ x^{-1}) - \nabla_{x_v \circ x^{-1}(x(u_0, v_0))}(x_u \circ x^{-1})$$

$$= \nabla_{x_u(u_0, v_0)}(x_v \circ x^{-1}) - \nabla_{x_v(u_0, v_0)}(x_u \circ x^{-1})$$

$$= x_{vu}(u_0, v_0) - x_{uv}(u_0, v_0), \qquad \text{as in Comment (i) on Section V.4,}$$

$$= 0, \qquad \text{since } x_{uv} = x_{vu}.$$

Hence

$$d\phi(x_u \circ x^{-1}, x_v \circ x^{-1}) = x_u \circ x^{-1}[\phi(x_v \circ x^{-1})] - x_v \circ x^{-1}[\phi(x_u \circ x^{-1})]$$

and hence considered as functions on the domain of the patch x

$$d\phi(x_u, x_v) = x_u[\phi(x_v \circ x^{-1})] - x_v[\phi(x_u \circ x^{-1})]$$

$$= \frac{\partial}{\partial u}[\phi(x_v \circ x^{-1} \circ x)] - \frac{\partial}{\partial v}[\phi(x_u \circ x^{-1} \circ x)]$$

$$= \frac{\partial}{\partial u}(\phi(x_v)) - \frac{\partial}{\partial v}(\phi(x_u)).$$

Summary

Notation

$\omega_{12}, \omega_{13}, \omega_{23}$	Page 248, line 3
θ_1, θ_2	Page 248, line −15

Definitions

(i)	Adapted frame field	Page 246, Definition 1.1
(ii)	Connection forms $(\omega_{12}, \omega_{13}, \omega_{23})$ on a surface	Page 248, line 3
(iii)	Dual 1-forms θ_1, θ_2 ·	Page 248, line −16

Results

(i) There is an adapted frame field on a region of M if and only if the region is orientable and there exists a nonvanishing tangent vector field on it. This is always possible on the image of a patch. Page 246, Lemma 1.2

(ii) For a tangent vector v tangent to M at p and S the shape operator

$$S(v) = \omega_{13}(v)E_1(p) + \omega_{23}(v)E_2(p).$$

Page 248, Corollary 1.5

(iii) The structural equations

$$(1)\quad \begin{cases} d\theta_1 = \omega_{12} \wedge \theta_2 \\ d\theta_2 = \omega_{21} \wedge \theta_1 \end{cases}$$ First structural equations

$$(2)\quad \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$$ Symmetry equation

$$(3)\quad d\omega_{12} = \omega_{13} \wedge \omega_{32}$$ Gauss equation

$$(4)\quad \begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{cases}$$ Codazzi equations

Page 249, Theorem 1.7

Techniques

(i)	Describing the shape operator in terms of the connection forms.	Page 248, Corollary 1.5
(ii)	Using the structural equations.	Page 249, Theorem 1.7

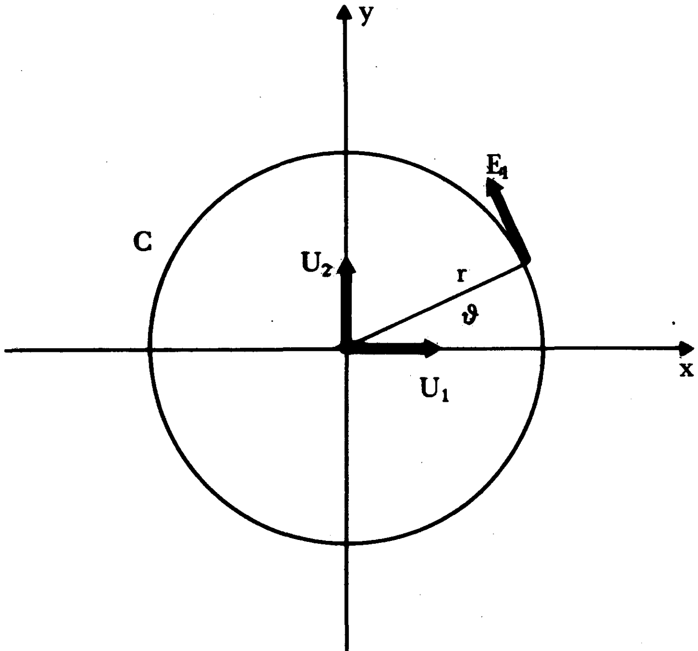
Exercises

Technique (i)

1. Page 250, Exercise 2.

Technique (ii)

2. (a) Let C be a circle of radius r in the xy plane and let the radius make an angle ϑ with the x axis. Let E_1 be the unit vector field along C that is in the xy plane, is tangent to C at each point of C and points in an anti-clockwise direction.



Prove that $E_1[\vartheta] = 1/r$.

(b) Page 250, Exercise 3. (Use the result of part (a) and compare the attitude matrix with that of the frame field of Fig. 6.3 on page 247 and Example 1.6 on page 248.)

(c) Page 250, Exercise 4.

Theory Exercise

3. Page 250, Exercise 1.

Solutions

1. Page 250, Exercise 2.

(a) We verify each of the structural equations in turn.

(1) $d\theta_1 = d(r\cos\varphi\, d\vartheta) = -r\sin\varphi\, d\varphi\, d\vartheta$, since $d(d\vartheta) = 0$ and r is constant on the sphere. Also

$\omega_{12} \wedge \theta_2 = \sin\varphi\, d\vartheta \wedge r d\varphi = r\sin\varphi\, d\varphi\, d\vartheta$.

$d\theta_1 = d(r\, d\varphi) = 0$, since r is constant, and

$\omega_{21} \wedge \theta_1 = -\sin\varphi\, d\vartheta \wedge r\cos\varphi\, d\vartheta = 0$, since $d\vartheta \wedge d\vartheta = 0$.

(2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$

$= (\cos\varphi\, d\vartheta) \wedge (r\cos\varphi\, d\vartheta) + d\varphi \wedge (r\, d\varphi) = 0$.

(3) $d\omega_{12} = d(\sin\varphi\, d\vartheta) = \cos\varphi\, d\varphi\, d\vartheta$;

$\omega_{13} \wedge \omega_{32} = (-\cos\varphi\, d\vartheta) \wedge d\varphi = \cos\varphi\, d\varphi\, d\vartheta$.

(4) $d\omega_{13} = d(-\cos\varphi\, d\vartheta) = \sin\varphi\, d\varphi\, d\vartheta;$

$\omega_{12} \wedge \omega_{23} = (\sin\varphi\, d\vartheta) \wedge (-d\varphi) = \sin\varphi\, d\varphi\, d\vartheta.$

$d\omega_{23} = d(-d\varphi) = 0;$

$\omega_{21} \wedge \omega_{13} = (-\sin\varphi\, d\vartheta) \wedge (-\cos\varphi\, d\vartheta) = 0.$

(b) $E_1[\vartheta] = d\vartheta[E_1] = \frac{1}{r\cos\varphi}\theta_1[E_1] = \frac{1}{r\cos\varphi};$

$E_2[\vartheta] = d\vartheta[E_2] = \frac{1}{r\cos\varphi}\theta_1[E_2] = 0;$

$E_1[\varphi] = d\varphi[E_1] = \frac{1}{r}\theta_2[E_1] = 0;$

$E_2[\varphi] = d\varphi[E_2] = \frac{1}{r}\theta_2[E_2] = \frac{1}{r}.$

(c) By Corollary VI.1.5,

$$\begin{aligned} S(v) &= \omega_{13}(v)E_1(p) + \omega_{23}(v)E_2(p) \\ &= -(\cos\varphi\, d\vartheta(v)E_1(p) + d\varphi(v)E_2(p)). \end{aligned}$$

Now if

$v = v_1 E_1(p) + v_2 E_2(p)$ then

$$\begin{aligned} d\vartheta(v) &= v_1\, d\vartheta(E_1(p)) + v_2\, d\vartheta(E_2(p)) \\ &= v_1 E_1(p)[\vartheta] + v_2 E_2(p)[\vartheta] \\ &= v_1/r\cos\varphi(p), \qquad \text{by part (b);} \end{aligned}$$

similarly

$$\begin{aligned} d\varphi(v) &= v_1 E_1(p)[\varphi] + v_2 E_2(p)[\varphi] \\ &= v_2/r. \end{aligned}$$

Hence

$$\begin{aligned} S(v) &= -\left(\frac{v_1\cos\varphi(p)}{r\cos\varphi(p)}E_1(p) + \frac{v_2}{r}E_2(p)\right) \\ &= -\frac{1}{r}(v_1 E_1(p) + v_2 E_2(p)) = -\frac{v}{r}. \end{aligned}$$

Hence the shape operator is given by the matrix

$$S = -\frac{1}{r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2(a). We can parametrize the circle C in such a way that the initial velocity is E_1 evaluated at the point corresponding to angle ϑ_0; that is,
$E_1 = -\sin\vartheta_0\, U_1 + \cos\vartheta_0\, U_2.$

We need

$$\alpha : t \longmapsto r\cos(\vartheta_0 + \tfrac{t}{r})\, U_1 + r\sin(\vartheta_0 + \tfrac{t}{r})\, U_2.$$

Hence

$$\begin{aligned} E_1[\vartheta] &= \frac{d}{dt}(\vartheta(\alpha))\Big|_{t=0} \\ &= \frac{d}{dt}(\vartheta_0 + \frac{t}{r})\Big|_{t=0} = \frac{1}{r}. \end{aligned}$$

2(b). Page 250, Exercise 3.

Replacing r by $(R + r\cos\varphi)$ in the previous part of the exercise we find that

$$E_1[\vartheta] = 1/(R + r\cos\varphi).$$

Since E_2 lies in a plane ϑ = constant

$$E_2[\vartheta] = 0.$$

Hence $$d\vartheta[E_1] = 1/(R + r\cos\varphi)$$

$$d\vartheta[E_2] = 0$$

and consequently $$\theta_1 = (R + r\cos\varphi)\,d\vartheta.$$

Similarly $$\theta_2 = r d\varphi.$$

Comparing Fig. 6.4 with Fig. 6.3 on page 247 we see that the attitude matrix for the frame field on the torus is the same as that for the frame field on the sphere. Hence the connection forms are the same. Those for the sphere were obtained on page 249 and so

$$\omega_{12} = \sin\varphi\, d\vartheta, \quad \omega_{13} = -\cos\varphi\, d\vartheta, \quad \omega_{23} = -d\varphi.$$

The structural equations are verified as for the spherical frame field.

2(c). Page 250, Exercise 4.

$$S(V) = \omega_{13}(V)E_1 + \omega_{23}(V)E_2.$$

$$S(V) = -\cos\varphi\, d\vartheta(V)E_1 - d\varphi(V)E_2$$

$$= -\frac{\cos\varphi}{(R + r\cos\varphi)}\,\theta_1(V)E_1 - \frac{1}{r}\theta_2(V)E_2.$$

$$S(E_1) = -(\cos\varphi/(R + r\cos\varphi))\,E_1.$$

$$S(E_2) = -(1/r)E_2.$$

Hence $k_1 = -\cos\varphi/(R + r\cos\varphi)$ and $k_2 = -1/r$.

3. Page 250, Exercise 1.

(a) $$\alpha'' = T' = \nabla_T T, \quad \text{since } T = \alpha',$$

$$= \nabla_T E_1, \quad \text{since } T = E_1,$$

$$= \omega_{12}(T)E_2 + \omega_{13}(T)E_3.$$

Now α is a geodesic if and only if α'' is normal to the surface, which happens if and only if $\omega_{12}(T) = 0$.

(b) If we let T, V, U be the restrictions of E_1, E_2, E_3 to α then

$$T' = \omega_{12}(T)V + \omega_{13}(T)U,$$

and so by Exercise V.5.7

$$g = \omega_{12}(T) \text{ and } k = \omega_{13}(T).$$

Again from Exercise V.5.7

$$t = S(T)\cdot V$$

$$= (\omega_{13}(T)T + \omega_{23}(T)V)\cdot V, \qquad \text{by Corollary 1.5,}$$

$$= \omega_{23}(T).$$

VI.2 FORM COMPUTATIONS

Introduction

This section follows on directly from the previous section. It uses the structural equations to obtain formulas for the mean and Gaussian curvature, introduced in Section V.3, and derives another form of the Codazzi equations when an adapted frame field can be chosen using principal directions.

The equation involving the Gaussian curvature is of fundamental importance in establishing, in Section VI.5, that Gaussian curvature belongs to an 'intrinsic' geometry.

READ: Section VI.2 (pages 251-255).

Supplementary Comments

(i) *Page 252: Lemma 2.2(2), the trace formula*

$$(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23})(E_1, E_2)$$

$$= \omega_{13}(E_1)\theta_2(E_2) - \omega_{13}(E_2)\theta_2(E_1) + \theta_1(E_1)\omega_{23}(E_2) - \theta_1(E_2)\omega_{23}(E_1)$$

$$= \omega_{13}(E_1) + \omega_{23}(E_2), \qquad \text{since } \theta_i(E_j) = \delta_{ij},$$

$$= \text{trace } S = 2H,$$

since the trace is found by adding the elements on the diagonal of any matrix representing the linear transformation.

(ii) *Page 253: Corollary 2.4, line -4 of the proof*

$$d\omega_{12}(E_1, E_2) = (df_1 \wedge \theta_1 + df_2 \wedge \theta_2 + f_1\omega_{12} \wedge \theta_2 + f_2\omega_{21} \wedge \theta_1)(E_1, E_2)$$

$$= df_1(E_1)\theta_1(E_2) - df_1(E_2)\theta_1(E_1) + df_2(E_1)\theta_2(E_2) - df_2(E_2)\theta_2(E_1)$$

$$+ f_1\omega_{12}(E_1)\theta_2(E_2) - f_1\omega_{12}(E_2)\theta_2(E_1) + f_2\omega_{21}(E_1)\theta_1(E_2) - f_2\omega_{21}(E_2)\theta_1(E_1)$$

$$= -df_1(E_2) + df_2(E_1) + f_1\omega_{12}(E_1) - f_2\omega_{21}(E_2).$$

(iii) *Page 253: line -7*

$$\omega_{12}(E_1) = \sin\varphi\, d\vartheta(E_1) = \sin\varphi\, E_1[\vartheta] = \sin\varphi/r\cos\varphi, \qquad \text{by Exercise VI.1.2,}$$

$$= \tan\varphi/r.$$

$$\omega_{12}(E_2) = \sin\varphi\, E_2[\vartheta] = 0.$$

(iv) *Page 253: line -3* This uses the chain rule $V[h(f)] = h'(f)\,V[f]$, where V is a tangent vector field on M, f is a function from M to R and h is a function from R to itself.

(v) *Page 254: Lemma 2.6, line 7 of the proof* If $V_1 = aF_1 + bF_2$ is an eigenvector corresponding to the eigenvalue k_1 then

$$\begin{pmatrix} S_{11} - k_1 & S_{12} \\ S_{12} & S_{22} - k_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so

$$a(S_{11} - k_1) + bS_{12} = 0.$$

Choosing $a = S_{12}$ gives $b = k_1 - S_{11}$ and so V_1 is as stated; and V_2 follows similarly.

(vi) *Page 254: Lemma 2.6, line −5 of the proof* $S(F_1) \cdot F_2$ is zero if and only if $S(F_1)$ is perpendicular to F_2, that is collinear to F_1. Hence $S(F_1) \cdot F_2$ is zero if and only if F_1 is a principal vector, which possibility we have excluded by considering only a small neighbourhood about p.

Summary

Definitions

(i)	Tangent frame field	Page 251, Section VI.2, line 4
(ii)	Principal frame field	Page 254, Definition 2.5

Results

(i)	For 1-forms ϕ and ψ and 2-forms μ and ν, $\phi = \psi$ if and only if $\phi(E_1) = \psi(E_1)$ and $\phi(E_2) = \psi(E_2)$ and $\mu = \nu$ if and only if $\mu(E_1, E_2) = \nu(E_1, E_2)$.	Page 251, lines −13 and −11
(ii)	$\theta_i(E_j) = \delta_{ij}$.	Page 251, line −7
(iii)	$\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$. $\mu = \mu(E_1, E_2)\theta_1 \wedge \theta_2$.	Page 251, Lemma 2.1
(iv)	$\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2$. $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$.	Page 252, Lemma 2.2
(v)	$d\omega_{12} = -K\theta_1 \wedge \theta_2$.	Page 252, Corollary 2.3
(vi)	$K = E_2[\omega_{12}(E_1)] - E_1[\omega_{12}(E_2)] - \omega_{12}(E_1)^2 - \omega_{12}(E_2)^2$.	Page 253, Corollary 2.4
(vii)	If p is nonumbilic then there exists a principal frame field on some neighbourhood of p in M.	Page 254, Lemma 2.6
(viii)	If E_1, E_2, E_3 is a principal frame field then $\omega_{13} = k_1\theta_1, \qquad \omega_{23} = k_2\theta_2$.	Page 254, line −4
(ix)	If E_1, E_2, E_3 is a principal frame field then $E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$ $E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$.	Page 255, Lemma 2.7

Techniques

(i)	Expressing forms in terms of θ_1, θ_2 and $\theta_1 \wedge \theta_2$.	Page 251, Lemma 2.1
(ii)	Using the structural equations to evaluate exterior derivatives of forms described in terms of $\theta_1, \theta_2, \omega_{12}, \omega_{13}$ and ω_{23}.	Page 249, Theorem 1.7
(iii)	Using the second structural equation to find the Gaussian curvature: $d\omega_{12} = -K\theta_1 \wedge \theta_2$.	Page 252, Corollary 2.3

Exercises

Techniques (i) and (ii)

1. Page 256, Exercise 2.
2. Page 256, Exercise 3.

Technique (iii)

3. Use the formulas of Exercise 3 on page 250 to calculate the Gaussian curvature of the torus.

Solutions

1. Page 256, Exercise 2.

Since $h_i = X \cdot E_i$, then for any vector field V on M

$$\begin{aligned} dh_i(V) &= V[h_i] = V[X \cdot E_i] \\ &= \nabla_V X \cdot E_i + X \cdot \nabla_V E_i, \qquad \text{by Corollary II.5.4(4),} \\ &= V \cdot E_i + X \cdot \nabla_V E_i. \end{aligned}$$

Hence, by the definition of the dual 1-forms and the connection forms,

$$\begin{aligned} dh_1(V) &= \theta_1(V) + X \cdot (\omega_{12}(V)E_2 + \omega_{13}(V)E_3) \\ &= \theta_1(V) + \omega_{12}(V)\,h_2 + \omega_{13}(V)\,h_3. \end{aligned}$$

Since this is true for all tangent vector fields V it follows that

$$dh_1 = \theta_1 + \omega_{12}h_2 + \omega_{13}h_3$$

and similarly

$$dh_2 = \theta_2 + \omega_{21}h_1 + \omega_{23}h_3.$$

2. Page 256, Exercise 3.

(a) By definition $\psi(V) = X \cdot V \times E_3$.

Now
$$\begin{aligned} \psi &= \psi(E_1)\theta_1 + \psi(E_2)\theta_2 \\ &= (X \cdot (E_1 \times E_3))\theta_1 + (X \cdot (E_2 \times E_3))\theta_2 \\ &= -(X \cdot E_2)\theta_1 + (X \cdot E_1)\theta_2 \\ &= -h_2\theta_1 + h_1\theta_2. \end{aligned}$$

Hence

$$\begin{aligned} d\psi &= -dh_2 \wedge \theta_1 - h_2 d\theta_1 + dh_1 \wedge \theta_2 + h_1 d\theta_2 \\ &= -(\theta_2 + \omega_{21}h_1 + \omega_{23}h_3) \wedge \theta_1 - h_2\omega_{12} \wedge \theta_2 \\ &\quad + (\theta_1 + \omega_{12}h_2 + \omega_{13}h_3) \wedge \theta_2 + h_1\,\omega_{21} \wedge \theta_1, \qquad \text{by Exercise 1,} \\ &= 2\theta_1 \wedge \theta_2 + h_3(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23}) \\ &= 2\theta_1 \wedge \theta_2 + 2h_3 H\theta_1 \wedge \theta_2, \qquad \text{by Lemma VI.2.2(2),} \\ &= 2(1 + hH)\theta_1 \wedge \theta_2. \end{aligned}$$

where $h_3 = h$ is the support function.

(b)
$$\begin{aligned}
\zeta(V) &= \psi(S(V)) \\
&= \psi(\omega_{13}(V)E_1 + \omega_{23}(V)E_2), \qquad \text{by Corollary VI.1.5,} \\
&= X\cdot(\omega_{13}(V)E_1 + \omega_{23}(V)E_2)\times E_3 \\
&= -\omega_{13}(V)(X\cdot E_2) + \omega_{23}(V)(X\cdot E_1).
\end{aligned}$$

Hence

$$\zeta = -h_2\,\omega_{13} + h_1\,\omega_{23}$$

and

$$\begin{aligned}
d\zeta &= -\,dh_2 \wedge \omega_{13} - h_2\,d\omega_{13} + dh_1 \wedge \omega_{23} + h_1\,d\omega_{23} \\
&= -\,(\theta_2 + \omega_{21}h_1 + \omega_{23}h_3) \wedge \omega_{13} - h_2(\omega_{12} \wedge \omega_{23}) \\
&\quad + (\theta_1 + \omega_{12}h_2 + \omega_{13}h_3) \wedge \omega_{23} + h_1(\omega_{21} \wedge \omega_{13}), \qquad \text{by Exercise 1,} \\
&= \omega_{13}\wedge\theta_2 + \theta_1\wedge\omega_{23} + 2h_3(\omega_{13}\wedge\omega_{23}) \\
&= 2(H + hK)\theta_1 \wedge \theta_2, \qquad \text{by Lemma VI.2.2 (1) and (2),}
\end{aligned}$$

where $h_3 = h$ is the support function.

3. $d\omega_{12} = d(\sin\varphi\, d\vartheta) = \cos\varphi\, d\varphi\, d\vartheta;$

$$\begin{aligned}
\theta_1\wedge\theta_2 &= r(R + r\cos\varphi)\, d\vartheta\, d\varphi \\
&= -r(R + r\cos\varphi)\, d\varphi\, d\vartheta.
\end{aligned}$$

Since $d\omega_{12} = -K\theta_1\wedge\theta_2$ it follows that

$$K = \frac{\cos\varphi}{r(R + r\cos\varphi)}\,.$$

VI.3 SOME GLOBAL THEOREMS

Introduction

This section is of a rather different nature to those we have just studied. In this section we study how the shape operator determines the shape of the surface in some restricted cases:

> If the shape operator is identically zero the surface belongs to a plane;
>
> If the surface is all umbilic then it is part of a plane or a sphere.

To go further you will need to be familiar with the concepts of *compactness* and *connectedness*. If you have not met compactness do not worry since the work depending on it is contained in the Optional Text. If you have met compactness and connectedness you should find these results interesting.

Connectedness and Compactness were defined in M202 in *Mendelson: Introduction To Topology,* Definition IV.2.2 and Definition V.2.4 respectively. In M332 they are given by the Definitions on pages 77 and 89 of *Unit 2, Continuous Functions.*

You will need to use the following two results:

(1) If a compact surface M is contained in a connected surface N then M = N.

(Sketch of proof: Since M is compact it is closed. Since M is a surface each point has a neighbourhood homeomorphic to an open disc and hence it is open in the subspace topology. The result now follows from the definition of connectedness.)

(2) If f is a continuous real valued function on a compact surface then f is bounded and attains its bounds.

(This was proved in *Unit M332 2, Continuous Functions,* pages 90-92, Theorems 10-12. It also appears as *Mendelson* Theorems V.2.9 and V.3.5.)

With these results you can read the Optional Text, where we prove the following important theorem:

> If the Gaussian curvature is constant and the surface is compact then the surface is a sphere.

READ: Section VI.3 (page 256 - page 259, line 7).

Comment

(i) *Page 258: line 3* If α is any curve in M then

$$dK(\alpha') = \alpha'[K] = (K \circ \alpha)' = 0, \qquad \text{by Lemma I.4.6.}$$

Hence $K \circ \alpha$ is constant; that is, K is constant along the curve α. Since the surface is connected any two points can be connected by a curve and hence the Gaussian curvature K is the same at any point of the surface.

Supplementary Comments

(i) *Page 256 : line −2* For any curve α, $S(\alpha') = -U'$, where the normal vector field U is restricted to α. In this case $U = E_3$ and $S(\alpha') = 0$. Hence $E_3' = 0$ and so the vector field E_3 is parallel along α. Since M is connected any two points can be connected by a curve and so E_3 is parallel on all of M.

(ii) *Page 256: line −1* The plane through p orthogonal to the direction E_3 consists of all those points r such that $(r - p) \cdot E_3 = 0$.

(iii) *Page 257: line 6* $\alpha' \cdot E_3 = 0$ since α' is a tangent vector and E_3 is the unit normal.

(iv) *Page 257: line -2* Theorem VI.2.7 gives

$$\begin{aligned} E_1[k] = E_1[k_2] &= (k_1 - k_2)\,\omega_{12}(E_2) \\ &= (k-k)\omega_{12}(E_2) = 0, \end{aligned}$$

and similarly $E_2[k] = 0$.

(v) *Page 258: line 1* $dk = dk(E_1)\theta_1 + dk(E_2)\theta_2 = 0$.

(vi) *Page 259: the end of the proof of Theorem 3.3* If we have chosen the outward normal for E_3 it follows that k is the negative square root of K, while if we have chosen the inward normal for E_3 it follows that k is the positive square root of K.

Optional Text

READ: Section VI.3 (page 259, line 8 to page 263).

Supplementary Comments

(i) *Page 262: first paragraph of the proof of Theorem 3.7*

$$\begin{aligned} H^2 - K &= \left(\frac{k_1 + k_2}{2}\right)^2 - k_1 k_2 = \frac{k_1^2 + 2k_1k_2 + k_2^2 - 4k_1k_2}{4} \\ &= \frac{k_1^2 - 2k_1k_2 + k_2^2}{4} = \frac{(k_1 - k_2)^2}{4} \geqslant 0. \end{aligned}$$

$H^2 - K = 0 \Rightarrow k_1 = k_2$ and so M is all-umbilic.

(ii) *Page 262: last four lines of the proof of Theorem 3.7*

$$(k_1 + k_2)^2 = (k_1 - k_2)^2 + 4K.$$

Suppose $k_1 > k_2 > 0$: then since K is constant $k_1 + k_2$ and $k_1 - k_2$ attain local maxima together, at which points $k_1 = \frac{1}{2}((k_1 + k_2) + (k_1 - k_2))$ does also. Since $k_2 = K/k_1$ the value of k_2 is a local minimum there.

Summary

Results

(i)	If the shape operator is identically zero then M is part of a plane.	Page 256, Theorem 3.1
(ii)	If M is all-umbilic it has constant Gaussian curvature $K \geqslant 0$.	Page 257, Lemma 3.2
(iii)	If M is all-umbilic and $K > 0$ then M is part of a sphere of radius $1/\sqrt{K}$.	Page 258, Theorem 3.3
(iv)	A compact all-umbilic surface is an entire sphere.	Page 259, Corollary 3.4
(v)	If M is a compact surface there is a point at which the Gaussian curvature K is strictly positive.	Page 259, Theorem 3.5
(vi)	If M is compact with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$.	Page 262, Theorem 3.7

Technique

(i) Application of the above results.

Exercise

Technique (i)

1. Page 263, Exercise 1.

Solution

1. Page 263, Exercise 1.

Since M is flat and minimal

$$K = k_1 k_2 = 0 \quad \text{and} \quad H = \tfrac{1}{2}(k_1 + k_2) = 0$$

and so $k_1 = k_2 = 0$. Hence $S = 0$ and M is part of a plane.

VI.4 ISOMETRIES AND LOCAL ISOMETRIES

Introduction

In Chapter III we defined the isometries of E^3 to be the distance-preserving mappings of E^3, and said that the geometry of E^3 consisted of those properties invariant under isometries of E^3. In order to make sense of the geometry of a surface we need the concept of "isometry of surfaces", and this section is devoted to adapting the ideas of Sections III.1 and III.2, that is the definition of isometries and the fact that their derivative maps preserve dot products of tangent vectors, to provide a definition of "isometry of surfaces" and explore some of its consequences.

Take care not to confuse "isometry of surfaces" with "isometry of E^3". An isometry of E^3 is a mapping from E^3 to E^3 which preserves lengths; an isometry of surfaces is a mapping F from a surface M to a surface N with certain nice properties : in general it is not possible to extend F so that it is defined on the whole of E^3, and so F cannot be the restriction of an isometry of E^3. The exact relationship between these two sorts of isometry will be explained in Section VI.8.

Of course, since isometries of surfaces are firstly mappings of surfaces, we need the results of Section IV.5 and the comments thereon. All that is needed from subsequent sections of O'Neill are the definitions of E, F and G on page 210 and the canonical parametrization of the catenoid from Section V.6. O'Neill also refers forward to Exercise 1 on page 269: this is a rewording of the following theorem of linear algebra, which is very similar to many results we used in Chapter III.

Theorem VI.4.A If V and W are 2-dimensional real vector spaces with inner products, which are written with dots in the usual way, and $T : V \longrightarrow W$ is a linear transformation, then the following conditions on T are logically equivalent; that is, if any one of them is true then all the rest are also true.

(a) T preserves dot products; that is,

$$T(v_1)\cdot T(v_2) = v_1 \cdot v_2$$

for all v_1, v_2 in V.

(b) T preserves lengths; that is,

$$\|T(v)\| = \|v\|$$

for all v in V.

(c) T preserves orthonormal bases; that is, if e_1, e_2 is any orthonormal basis for V then $T(e_1), T(e_2)$ is an orthonormal basis for W.

(c′) T preserves a particular orthonormal basis; that is, there is some particular orthonormal basis e_1, e_2 for V such that $T(e_1), T(e_2)$ is an orthonormal basis for W.

(d) There is some pair of linearly independent vectors v_1, v_2 in V such that

$$\|T(v_1)\| = \|v_1\|$$
$$\|T(v_2)\| = \|v_2\|$$
$$T(v_1)\cdot T(v_2) = v_1 \cdot v_2 .$$

Proof Conditions (a) and (b) were shown to be equivalent in *Unit M201 24, Orthogonal and Symmetric Transformations,* Theorem 1 (page 7) and the details of the proof of Theorem 2 on pages 10-11 of that unit show that conditions (a), (c) and (c′) are all equivalent. The remainder of the proof is contained in the following exercise.

1. Prove that (c′) ⇒ (d) and (d) ⇒ (b), where (b), (c′) and (d) denote the corresponding conditions of Theorem VI.4.A. This is sufficient to prove that condition (d) is equivalent to each of conditions (a), (b), (c), (c′).

We shall also need

Theorem VI.4.B If $T: V \longrightarrow W$ is an orthogonal transformation then T has an inverse $T^{-1}: W \longrightarrow V$, which is also orthogonal.

Proof The proof is given in the Correspondence Text, Part III, page 6, Comment (iii).

READ: Section VI.4 (pages 263-269).

Comments

(i) *Page 264: Definition 4.1* A curve segment α from p to q is a portion of a curve defined on a *closed* interval [a, b] such that $\alpha(a) = p$, $\alpha(b) = q$. The length of such a curve segment is the arc length from a to b, that is

$$\int_a^b \|\alpha'(t)\| \, dt.$$

Because all such lengths are non-negative, the set of appropriate "$L(\alpha)$" s is bounded below (by zero) and so does have a greatest lower bound.

(ii) *Page 264: Definition 4.2* We require that

$$F_*(v) \cdot F_*(w) = v \cdot w$$

whenever these expressions are meaningful. $F_*(v)$ and $F_*(w)$ are defined only when v and w are tangent vectors which are tangent to M. If v and w are two such vectors, $v \cdot w$ is defined if and only if v, w belong to the same tangent space $T_p(M)$: in this case $F_*(v)$ and $F_*(w)$ both belong to the tangent space $T_{F(p)}(M)$ and so $F_*(v) \cdot F_*(w)$ is defined. Thus we are requiring that

$$F_*(v) \cdot F_*(w) = v \cdot w$$

whenever v, w belong to the same tangent space $T_p(M)$ for some p in M.

Supplementary Comments

(i) *Page 265: line 1* This is the implication (a) ⇒ (b) of Theorem VI.4.A.

(ii) *Page 265: proof of Theorem 4.3* $F_*(\alpha') = (F(\alpha))'$ by the definition of F_*.

(iii) *Page 266: Lemma 4.5, and subsequently* O'Neill is using the letter F to denote both a mapping of surfaces and the function $x_u \cdot x_v$. This looks confusing, especially where both usages appear in close proximity, but it should always be clear from the context which meaning is intended.

(iv) *Page 266: proof of Lemma 4.5* This proof uses the equivalence (d) ⇔ (a) of Theorem VI.4.A, the fact that x_u and x_v are linearly independent at each point, and the result that

$$F_*(x_u) = (F(x))_u, \quad F_*(x_v) = (F(x))_v,$$

which was proved on page 161.

(v) *Page 267: Example 4.6(1)* This uses the equivalence (c′) ⇔ (a) of Theorem VI.4.A, as $U_1(p)$ and $U_2(p)$ form an orthonormal basis for $T_p(E^2)$.

(vi) *Page 267: Example 4.6(2)* E, F, G (for the helicoid with $b = 1$) were computed in Example V.4.3(1) on page 213; $\overline{E}$, $\overline{F}$, $\overline{G}$ (for the catenoid with a canonical parametrization) were computed in Lemma V.6.3 on page 238.

Summary

Notation

$L(\alpha)$	Page 264, Definition 4.1
$\rho(p, q)$	Page 264, Definition 4.1

Definitions

(i)	Length ($L(\alpha)$) of a curve segment α	Text, page 20, Comment (i)
(ii)	Intrinsic distance $\rho(p, q)$	Page 264, Definition 4.1
(iii)	Isometry of surfaces	Page 264, Definition 4.2
(iv)	Isometric surfaces	Page 265, line −8
(v)	Local isometry	Page 265, Definition 4.4
(vi)	Conformal mapping of surfaces	Page 268, Definition 4.7

Examples

(i)	A plane is locally isometric to a circular cylinder.	Page 267, Example 4.6(1)
(ii)	A helicoid is locally isometric to a catenoid.	Page 267, Example 4.6(2)

Results

(i)	Isometries of surfaces have inverse mappings; that is, they are diffeomorphisms.	Page 265, line 6
(ii)	Isometries of surfaces preserve intrinsic distance.	Page 265, Theorem 4.3
(iii)	Isometries preserve the speed and length of curves.	Page 265, line 14
(iv)	A local isometry is an isometry on sufficiently small neighbourhoods.	Page 266, line 7
(v)	A mapping $F : M \longrightarrow N$ is a local isometry if and only if $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$ for each patch x in M.	Page 266, Lemma 4.5
(vi)	Local isometries are conformal mappings with $\lambda(p) = 1$.	Page 268, line −2

(vii) Each of the following conditions is equivalent to F being a local isometry:

(a) F_* preserves dot products;

(b) F_* preserves lengths;

(c) F_* preserves tangent frames;

(d) F_* preserves $\|v\|$, $\|w\|$ and $v \cdot w$ for some pair of linearly independent tangent vectors v and w in $T_p(M)$ for each p.

Text, page 19,
Theorem VI.4.A and
Page 269, Exercise 1

Techniques

(i) Recognizing isometries of surfaces and local isometries.

Page 264, Definition 4.2 and
Page 265, Definition 4.4

(ii) Finding local isometries using Result (v).

Page 266, Lemma 4.5

Exercises

Technique (i)

2. Let C be the catenic cylinder, which is the image of the patch

$$x : (u, v) \longrightarrow (\sinh^{-1} u, (1 + u^2)^{\frac{1}{2}}, v) \qquad (u, v) \in E^2.$$

That is, C is a cylinder whose cross-section is a catenary (see Fig. 5.41 on page 239). Show that C is isometric to E^2 by showing that $\underset{\sim}{x}$ is an isometry of surfaces. (The derivative of $u \longmapsto \sinh^{-1} u$ is $u \longmapsto (1 + u^2)^{-\frac{1}{2}}$.)

Technique (ii)

3. Page 270, Exercise 5(a). You may find it helpful to look at Exercise V.5.11 on page 231, where the tangent surface of a curve was defined.

Theory Exercise

4. Prove that, for any diffeomorphism (that is, mapping with an inverse mapping) F of surfaces, $(F^{-1})_* = F_*^{-1}$. Deduce that the relation "is isometric to" is an equivalence relation on the class of all surfaces in E^3.

Optional Section – Conformal Mappings

O'Neill has barely introduced conformal mappings. They are not needed again in this course, but they are relevant to some other parts of Mathematics (M332 Complex Analysis, for example). If you would like to gain a deeper understanding of conformal mappings, work through the following exercises.

5. Write down conditions, in terms of $\lambda(p)$, analogous to (a), (b), (c), (c′), (d) of Theorem VI.4.A which are logically equivalent and which F_{*p} must satisfy for all p if F is to be conformal. For example, (b) becomes the definition of conformal:

   (b) For all v in $T_p(M)$

   $$\|F_*(v)\| = \lambda(p)\,\|v\|.$$

6. Page 270, Exercise 8. (ϑ is defined to be an angle between v and w if $\cos\vartheta = v \cdot w/\|v\|\,\|w\|$.)

7. Page 271, Exercise 14. Apply condition (d) of Exercise 5 to x_u, x_v for suitable patches x. You will need to use two patches.

   (a) All of Σ_0 except one semicircle from the north pole to the south pole is covered by the modified geographical patch

   $$x(u, v) = (\cos v \cos u, \cos v \sin u, 1 + \sin v), \qquad u \in (-\pi, \pi),\ v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

   This patch x, and the composite $P(x) = y$ (where P is stereographic projection) are given on page 162. The partial velocities x_u, x_v, y_u, y_v were computed in the solution to Exercise 3 on page 53 of the Text, Part IV. A similar patch, changing the domain of u to $(0, 2\pi)$, covers all of the missing semicircle except the south pole: the details of the computation are obviously the same and may be omitted.

   (b) The south pole is covered by the Monge patch

   $$x(u, v) = (u, v, 1 - (1 - u^2 - v^2)^{\frac{1}{2}}), \qquad u^2 + v^2 < 1.$$

   The values of the partial velocities of this patch x and the composite P(x) at the origin were computed in the solution to Exercise 3 on page 53 of the Text, Part IV.

Solutions

1. *(c′) ⇒ (d)* Let $v_1 = e_1$, $v_2 = e_2$, where e_1, e_2 is the particular orthonormal basis for V specified in (c′). Then v_1, v_2 are linearly independent, and

   $$\|T(v_1)\| = \|T(e_1)\| = 1 = \|e_1\| = \|v_1\|$$

   $$\|T(v_2)\| = \|T(e_2)\| = 1 = \|e_2\| = \|v_2\|$$

   $$T(v_1) \cdot T(v_2) = T(e_1) \cdot T(e_2) = 0 = e_1 \cdot e_2 = v_1 \cdot v_2$$

   (using (c′)).

   (d) ⇒ (b) Let v_1, v_2 be the particular vectors specified in (d). These are linearly independent and so span V, and thus any vector v in V can be written as a linear combination of v_1, v_2:

   $$v = av_1 + bv_2$$

   where a, b are real numbers. T is a linear transformation so

   $$\begin{aligned} T(v) &= T(av_1 + bv_2) \\ &= aT(v_1) + bT(v_2). \end{aligned}$$

Thus

$$\begin{aligned}\|T(v)\|^2 &= \|aT(v_1)+bT(v_2)\|^2\\ &= (aT(v_1)+bT(v_2))\cdot(aT(v_1)+bT(v_2))\\ &= a^2T(v_1)\cdot T(v_1)+2abT(v_1)\cdot T(v_2)+b^2T(v_2)\cdot T(v_2)\\ &= a^2\|T(v_1)\|^2+2abT(v_1)\cdot T(v_2)+b^2\|T(v_2)\|^2\\ &= a^2\|v_1\|^2+2abv_1\cdot v_2+b^2\|v_2\|^2 \qquad \text{(by (d))}\\ &= (av_1+bv_2)\cdot(av_1+bv_2)\\ &= \|av_1+bv_2\|^2\\ &= \|v\|^2.\end{aligned}$$

2. The patch $x : E^2 \longrightarrow C$ is onto by definition of C. Moreover

$$x : (u, v) \longmapsto (\sinh^{-1}u, (1+u^2)^{\frac{1}{2}}, v) \qquad (u, v) \in E^2$$

is one-to-one because the functions

$$u \longmapsto \sinh^{-1}u \qquad (u \in R)$$
$$v \longmapsto v \qquad (v \in R)$$

are both one-to-one. Thus we have to show just that x_* preserves dot products of tangent vectors. Following the method of Example VI.4.6(1), it suffices to show that $E = G = 1$, $F = 0$.

$$x_u = \left(\frac{1}{(1+u^2)^{\frac{1}{2}}},\ \frac{u}{(1+u^2)^{\frac{1}{2}}},\ 0\right)$$

$$x_v = (0,\ 0,\ 1)$$

so $E = x_u \cdot x_u = \dfrac{1+u^2}{1+u^2} = 1$, $F = x_u\cdot x_v = 0$, $G = x_v\cdot x_v = 1$,

and so x is an isometry.

3. Page 270, Exercise 5(a).

The tangent surface A to α is the image of the patch

$$x : (u, v) \longmapsto \alpha(u) + vT(u)$$

where T(u) is the vector such that

$$\alpha'(u) = T(u)_{\alpha(u)}.$$

The tangent surface B to β is parametrized by

$$y : (u, v) \longmapsto \beta(u) + v\overline{T}(u)$$

where

$$\beta'(u) = \overline{T}(u)_{\beta(u)}.$$

An obvious mapping from A to B is F, where F is defined by $F(x) = y$. By Lemma VI.4.5, F is a local isometry if $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$. Now

$$\begin{aligned}x_u &= \alpha'(u) + vT'(u)_{\alpha(u)}\\ &= (T(u) + v\kappa(u)N(u))_{\alpha(u)}\end{aligned}$$

where $N(u)_{\alpha(u)}$ is the principal normal to α at $\alpha(u)$, and

$$x_v = T(u)_{\alpha(u)}:$$

$$\text{thus } E = x_u \cdot x_u = T(u)\cdot T(u) + 2v\kappa(u)T(u)\cdot N(u) + (v\kappa(u))^2 N(u)\cdot N(u)$$
$$= 1 + 0 + (v\kappa(u))^2;$$
$$F = x_u \cdot x_v = T(u)\cdot T(u) + v\kappa(u)T(u)\cdot N(u)$$
$$= 1;$$
$$G = x_v \cdot x_v = T(u)\cdot T(u) = 1.$$

Since β has the same curvature function κ, a similar argument shows that

$$\overline{E} = 1 + (v\kappa(u))^2$$
$$\overline{F} = 1$$
$$\overline{G} = 1.$$

Hence $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$, and so F is an isometry.

4. By the definition of inverse mapping,

$$FF^{-1} = F^{-1}F = I.$$

Thus the chain rule gives

$$F_*(F^{-1})_* = (F^{-1})_*F_* = I_*.$$

What is I_*? Well, I_* is defined so that

$$I_*(\alpha') = (I(\alpha))'$$

for any tangent vector α' tangent to M. I is the identity map on M, and so $(I(\alpha))' = \alpha'$ and thus $I_*(\alpha') = \alpha'$: in other words I_* is the identity map on the tangent spaces to M. Thus $F_*(F^{-1})_*$ and $(F^{-1})_* F_*$ are both the identity map on the tangent spaces to M, and so $(F^{-1})_*$ is the inverse of F_*; that is,

$$(F^{-1})_* = (F_*)^{-1}.$$

Now, to show that the relation "is isometric to" is an equivalence relation we must show that it is (a) reflexive, (b) symmetric, and (c) transitive.

(a) If M is a surface in E^3, the identity map $I : M \longrightarrow M$ is clearly an isometry, so M is isometric to itself. Thus "is isometric to" is reflexive.

(b) If $F : M \longrightarrow N$ is an isometry of surfaces, we saw on page 265 that F has an inverse mapping $F^{-1} : N \longrightarrow M$. Since F^{-1} has F as an inverse mapping, F^{-1} must itself be one-to-one and onto. To show that F^{-1} is an isometry we need to show that $(F^{-1})_*$ preserves dot products. We have shown that $(F^{-1})_* = (F_*)^{-1}$. Now F_* preserves dot products; that is, each F_{*p} is an orthogonal transformation. Consequently Theorem VI.4.B tells us that each ${F_{*p}}^{-1}$ is an orthogonal transformation; that is ${F_*}^{-1}$ preserves dot products. Hence $(F^{-1})_*$ preserves dot products and so F^{-1} is an isometry.

(c) Suppose $F : M \longrightarrow N$ and $G : N \longrightarrow S$ are isometries of surfaces. Then F and G are both one-to-one and onto, so $GF : M \longrightarrow S$ is also one-to-one and onto. For tangent vectors v, w tangent to M at the same point,

$$\begin{aligned}((GF)_*v)\cdot((GF)_*w) &= G_*(F_*v)\cdot G_*(F_*w), && \text{by the chain rule,}\\ &= F_*v\cdot F_*w, && \text{because G is an isometry,}\\ &= v\cdot w, && \text{because F is an isometry.}\end{aligned}$$

Thus $GF : M \longrightarrow S$ is an isometry and so "is isometric to" is transitive.

5. (a) For all v_1, v_2 in $T_p(M)$

$$F_*(v_1)\cdot F_*(v_2) = (\lambda(p))^2 v_1\cdot v_2 .$$

(b) For all v in $T_p(M)$

$$\|F_*(v)\| = \lambda(p)\|v\|.$$

(c) For all frames e_1, e_2 in $T_p(M)$

$$\|F_*(e_1)\| = \|F_*(e_2)\| = \lambda(p)$$

$$F_*(e_1)\cdot F_*(e_2) = 0.$$

(c′) There is a particular frame e_1, e_2 in $T_p(M)$ such that

$$\|F_*(e_1)\| = \|F_*(e_2)\| = \lambda(p)$$

$$F_*(e_1)\cdot F_*(e_2) = 0.$$

(d) There is some pair of linearly independent vectors v_1, v_2 in $T_p(M)$ such that

$$\|F_*(v_1)\| = \lambda(p)\|v_1\|$$

$$\|F_*(v_2)\| = \lambda(p)\|v_2\|$$

$$F_*(v_1)\cdot F_*(v_2) = (\lambda(p))^2 v_1\cdot v_2 .$$

6. Page 270, Exercise 8.

If ϑ is an angle between v and w in $T_p(M)$ then

$$\cos\vartheta = \frac{v\cdot w}{\|v\|\,\|w\|}\cdot$$

If $F : M \longrightarrow N$ is conformal with scale factor λ then

$$\|F_*(v)\| = \lambda(p)\|v\|$$

$$\|F_*(w)\| = \lambda(p)\|w\|,$$

by definition of conformal, and

$$F_*(v)\cdot F_*(w) = (\lambda(p))^2 v\cdot w$$

by condition (a) in Exercise 5. Hence

$$\begin{aligned}\frac{F_*(v)\cdot F_*(w)}{\|F_*(v)\|\,\|F_*(w)\|} &= \frac{(\lambda(p))^2\, v\cdot w}{\lambda(p)\|v\|\;\lambda(p)\|w\|}\\ &= \frac{v\cdot w}{\|v\|\,\|w\|}\\ &= \cos\vartheta,\end{aligned}$$

and so ϑ is an angle between $F_*(v)$ and $F_*(w)$.

7. Page 271, Exercise 14.

(a) First we show that the mapping

$$P : (p_1, p_2, p_3) \longmapsto \frac{2}{2-p_3}(p_1, p_2)$$

is conformal on the image of the patch

$$x : (u, v) \longmapsto (\cos v \cos u, \cos v \sin u, 1 + \sin v),$$
$$(u \in (-\pi, \pi), v \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})).$$

Let $y = P(x)$. Then, since $P_*(x_u) = y_u$, $P_*(x_v) = y_v$, by condition (d) of Exercise 5 we have to show that

$$y_u \cdot y_u = \lambda^2 x_u \cdot x_u$$
$$y_u \cdot y_v = \lambda^2 x_u \cdot x_v$$
$$y_v \cdot y_v = \lambda^2 x_v \cdot x_v$$

for some λ depending on the point p. Now, we saw on page 53 of Text, Part IV that

$$x_u = (-\cos v \sin u, \cos v \cos u, 0)$$
$$x_v = (-\sin v \cos u, -\sin v \sin u, \cos v)$$
$$y_u = \frac{2\cos v}{1-\sin v}(-\sin u, \cos u)$$
$$y_v = \frac{2}{1-\sin v}(\cos u, \sin u)$$

and so

$$x_u \cdot x_u = \cos^2 v, \quad x_u \cdot x_v = 0, \quad x_v \cdot x_v = 1$$
$$y_u \cdot y_u = \frac{4\cos^2 v}{(1-\sin v)^2}, \quad y_u \cdot y_v = 0, \quad y_v \cdot y_v = \frac{4}{(1-\sin v)^2}.$$

Thus we have the required result with

$$\lambda(x(u, v)) = \frac{2}{1 - \sin v}.$$

If $p = x(u, v)$ then $P(p) = \frac{2\cos v}{1-\sin v}(\cos u, \sin u)$, so

$$\begin{aligned}1 + \frac{\|P(p)\|^2}{4} &= 1 + \frac{\cos^2 v}{(1-\sin v)^2}\\ &= \frac{1 - 2\sin v + \sin^2 v + \cos^2 v}{(1-\sin v)^2}\\ &= \frac{2(1-\sin v)}{(1-\sin v)^2}\\ &= \frac{2}{1-\sin v}\\ &= \lambda(p),\end{aligned}$$

as required.

Now we show that P is conformal at the south pole by using the Monge patch

$$x(u, v) = (u, v, 1 - (1 - u^2 - v^2)^{\frac{1}{2}}), \qquad u^2 + v^2 < 1.$$

Again we let $y = P(x)$. We saw on page 53 of Text, Part IV that, at the south pole,

$$x_u(0, 0) = (1, 0, 0)$$
$$x_v(0, 0) = (0, 1, 0)$$
$$y_u(0, 0) = (1, 0)$$
$$y_v(0, 0) = (0, 1)$$

and so

$$x_u \cdot x_u = 1, \quad x_u \cdot x_v = 0, \quad x_v \cdot x_v = 1$$

$$y_u \cdot y_u = 1, \quad y_u \cdot y_v = 0, \quad y_v \cdot y_v = 1$$

Thus we can take $\lambda(0) = 1$ to satisfy condition (d). Then

$$1 + \frac{\|P(0)\|^2}{4} = 1 + 0 = \lambda(0)$$

and so

$$\lambda(p) = 1 + \frac{\|P(p)\|^2}{4}$$

for all p in Σ_0.

Since we have found a positive value of λ for all p in Σ_0, P is conformal.

VI.5 INTRINSIC GEOMETRY OF SURFACES IN E^3

Introduction

This section follows on directly from Section VI.4. It also uses the function F^*, which was defined in Section IV.5, and the connection form computations from Section VI.2.

The purpose of this section is to prove Gauss' *theorema egregium*, that the Gaussian curvature of a surface in E^3 is left invariant by isometries. This is a very important result, which immediately gives us a simple criterion for telling when two surfaces are not isometric. The proof of the theorem is a little technical: we do not expect you to be able to reproduce it from memory.

READ: Section VI.5 (pages 271-275).

Comment

(i) *Page 272 : statement of Lemma 5.1* This is really saying that ω_{12} is the unique 1-form that satisfies

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

and that ω_{21} is given by

$$\omega_{21} = -\omega_{12}.$$

Supplementary Comments

(i) *Page 272 : proof of Lemma 5.1*

$$d\theta_1(E_1, E_2) = (\omega_{12} \wedge \theta_2)(E_1, E_2),$$ by the first structural equation,

$$= \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1),$$ by the definition of wedge product,

$$= \omega_{12}(E_1)\cdot 1 - \omega_{12}(E_2)\cdot 0,$$ by the definition of θ_2,

$$= \omega_{12}(E_1)$$

and

$$d\theta_2(E_1, E_2) = (\omega_{21} \wedge \theta_1)(E_1, E_2)$$

$$= \omega_{21}(E_1)\theta_1(E_2) - \omega_{21}(E_2)\theta_1(E_1)$$

$$= -\omega_{21}(E_2)$$

$$= \omega_{12}(E_2).$$

(ii) *Page 272 : end of Remark 5.2* To show that ω_{12} defined by

$$\omega_{12}(E_i) = d\theta_i(E_1, E_2)$$

satisfies the structural equations

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

it suffices to check these equations for the pair (E_1, E_2) (by Lemma VI.2.1).

Now

$$\begin{aligned}(\omega_{12} \wedge \theta_2)(E_1, E_2) &= \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1), && \text{by definition of th} \\ &&& \text{wedge product,} \\ &= \omega_{12}(E_1), && \text{since } \theta_2(E_i) = \delta_{i2}, \\ &= d\theta_1(E_1, E_2), && \text{by the new definition of } \omega_{12},\end{aligned}$$

so

$$\omega_{12} \wedge \theta_2 = d\theta_1.$$

Similarly

$$\begin{aligned}(-\omega_{12} \wedge \theta_1)(E_1, E_2) &= -\omega_{12}(E_1)\theta_1(E_2) + \omega_{12}(E_2)\theta_1(E_1) \\ &= \omega_{12}(E_2) \\ &= d\theta_2(E_1, E_2), && \text{by the new definition of } \omega_{12};\end{aligned}$$

so

$$-\omega_{12} \wedge \theta_1 = d\theta_2.$$

Summary

Definitions

(i)	Tangent frame field	Page 271, line -4
(ii)	Isometric invariant	Page 271, line 3 of the section
(iii)	Intrinsic geometry	

Results

(i)	ω_{12} is the only 1-form that satisfies $d\theta_1 = \omega_{12} \wedge \theta_2$ $d\theta_2 = -\omega_{12} \wedge \theta_1$.	Page 272, Lemma 5.1
(ii)	$\omega_{12}(E_i) = -\omega_{21}(E_i) = d\theta_i(E_1, E_2)$.	Page 272, Remark 5.2
(iii)	If F is an isometry, E_1, E_2 a tangent frame field, and $F_*(E_i) = \bar{E}_i$, then $\theta_i = F^*\bar{\theta}_i$ $\omega_{12} = F^*\bar{\omega}_{12}$.	Page 273, Lemma 5.3
(iv)	Gaussian curvature is an isometric invariant.	Page 273, Theorem 5.4
(v)	Local isometries preserve Gaussian curvature.	Page 274, line -8
(vi)	Geodesics are isometric invariants.	Text, page 31, Exercise 1
(vii)	Shape operators, principal directions, principal curvatures, mean curvature, principal curves, asymptotic curves, and the forms ω_{13}, ω_{23} are not isometric invariants.	Text, page 31, Exercise 2

Techniques

(i)	Deciding which properties of a surface are isometric invariants.	Page 271, line 3 of the section
(ii)	Using a knowledge of isometric invariants to find properties of surfaces isometric to a known surface.	Page 274, line -7
(iii)	Using a knowledge of isometric invariants to show that two surfaces are not isometric.	Page 275, line 2

Exercises

Technique (i)

1. Page 275, Exercise 1. (HINT: Since we have seen, on page 228, that non-constant geodesics are all constant-speed reparametrizations of unit-speed geodesics, and since isometries preserve the speed of curves, it is sufficient to consider the case when α has unit speed and use Exercise VI.1.1.)

2. Use Example VI.4.6 (1) on page 267 to deduce that principal vectors, principal curvatures, mean curvatures, shape operators, principal curves, asymptotic curves and the 1-forms ω_{13} and ω_{23} are not isometric invariants.

3. Page 275, Exercise 3. (If you have done M202 or M332 you will appreciate that the truth of the first sentence can be proved as follows:

   $K : M \longrightarrow R$ is a continuous function. Since M is a connected surface and the continuous image of a connected space is also connected, K(M) must be connected. However, $K(M) \subset R$, and the only connected subsets of R are the intervals.)

Technique (ii)

4. Page 275, Exercise 2. Omit the generalization.

Technique (iii)

5. Page 275, Exercise 4.

Solutions

1. Page 275, Exercise 1.

   We wish to show that if $F : M \longrightarrow N$ is a local isometry and α is a unit-speed geodesic in M then $F(\alpha)$ is a geodesic in N. Following the hint to use Exercise VI.1.1 we choose a tangent frame field E_1, E_2 on a region of M containing α such that E_1 restricted to α is its unit tangent T; that is, the value of the vector field E_1 at the point $\alpha(t)$ is the unit tangent T(t), so that

   $$E_1(\alpha(t)) = T(t) = \alpha'(t)$$

   for all t in the domain of α. Because we have chosen E_1 in this way, part (a) of Exercise VI.1.1 tells us that $\omega_{12}(T) = 0$ because α is a geodesic.

Let $\bar{E}_1 = F_*(E_1)$, $\bar{E}_2 = F_*(E_2)$, $\bar{\alpha} = F(\alpha)$, $\bar{T} = \bar{\alpha}'$. Then, as in Lemma VI.5.3, $\bar{E}_1$, $\bar{E}_2$ is a tangent frame field on N and

$$\omega_{12} = F^*(\bar{\omega}_{12}).$$

Moreover, for t in the domain of α,

$$\begin{aligned} \bar{E}_1(\bar{\alpha}(t)) &= \bar{E}_1(F(\alpha(t))) \\ &= F_*(E_1(\alpha(t))) \\ &= F_*(\alpha'(t)) \\ &= (F(\alpha))'(t), && \text{by definition of } F_*, \\ &= \bar{\alpha}'(t) \\ &= \bar{T}(t) \end{aligned}$$

so $\bar{E}_1$ coincides with $\bar{T}$ along $\bar{\alpha}$. Now, $\bar{\alpha}$ is also a unit-speed curve, because isometries preserve the speeds of curves, and we have just shown that $\bar{E}_1$ restricted to $\bar{\alpha}$ coincides with its unit tangent $\bar{T}$. Thus the conditions in Exercise VI.1.1 are satisfied by $\bar{\alpha}$, $\bar{E}_1$, $\bar{E}_2$ and so we see that $\bar{\alpha}$ is a geodesic in N if and only if $\bar{\omega}_{12}(\bar{T}) = 0$. Now

$$\begin{aligned} \bar{\omega}_{12}(\bar{T}) &= \bar{\omega}_{12}(\bar{\alpha}') \\ &= \bar{\omega}_{12}((F(\alpha))') \\ &= \bar{\omega}_{12}(F_*(\alpha')) \\ &= F^*(\bar{\omega}_{12})(\alpha'), && \text{by definition of } F^*, \\ &= \omega_{12}(\alpha'), && \text{by Lemma VI.5.3,} \\ &= \omega_{12}(T) \\ &= 0, && \text{because } \alpha \text{ is a geodesic.} \end{aligned}$$

Hence $\bar{\alpha}$ is a geodesic in N.

2. We wish to show that certain functions on E^2 and types of curve in E^2 are not "preserved" when mapped to the cylinder $M : x^2 + y^2 = r^2$ by the local isometry

$$x : (u, v) \longmapsto (r\cos(u/r), r\sin(u/r), v).$$

Since each small enough open set in M is isometric to a region in E^2, this will show that these functions and curves are not isometric invariants.

A function f on E^2 is said to be "preserved by x" if the "corresponding function" $\bar{f}$ on M satisfies

$$f = \bar{f}(x).$$

$$\begin{tikzcd} E^2 \arrow[rr, "x"] \arrow[dr, "f"'] & & M \arrow[dl, "\bar{f}"] \\ & R & \end{tikzcd}$$

Firstly take f to be each of the principal curvature functions, k_1 and k_2. The principal curvatures of E^2 are zero everywhere, so

$$k_1(p) = k_2(p) = 0$$

for all p in E^2. The principal curvatures of M are $0, -1/r$ everywhere so

$$\bar{k}_1(q) = 0, \bar{k}_2(q) = -1/r$$

for all q in M : in particular

$$\bar{k}_2(x(p)) = -1/r$$

for all p in E^2. Hence $\bar{k}_2(x)$ is not equal to k_1 or k_2 and so x does not preserve the pair of principal curvatures. It is true that in this case x preserves one of the principal curvatures, but this does not happen in general.

A similar treatment works for the mean curvature function H.

Now

$$H(p) = 0$$

for all p in E^2, and

$$\bar{H}(q) = -1/2r$$

for all q in M, so in particular

$$\bar{H}(x(p)) = -1/2r$$

for all p in E^2. Hence $\bar{H}(x)$ is not equal to H, and so x does not preserve mean curvatures.

Take $E_1 = U_1, E_2 = U_2$ on E^2, and $\bar{E}_i = x_*(E_i)$. Then ω_{13} is preserved by x if

$$\omega_{13} = \bar{\omega}_{13}(x_*).$$

$$\begin{tikzcd}
T_p(E^2) \arrow[rr, "x_*"] \arrow[dr, "\omega_{13}"'] & & T_{x(p)}(M) \arrow[dl, "\bar{\omega}_{13}"] \\
& R &
\end{tikzcd}$$

We can find the 1-forms ω_{13}, $\bar{\omega}_{13}$ from Corollary VI.1.5 :

$$S = \omega_{13}\, E_1 + \omega_{23}\, E_2$$

$$\bar{S} = \bar{\omega}_{13}\, \bar{E}_1 + \bar{\omega}_{23}\, \bar{E}_2.$$

On E^2, S is zero and so $\omega_{13} = \omega_{23} = 0$; on M, $S(\bar{E}_1) = -\bar{E}_1/r$ and so $\bar{\omega}_{13}(\bar{E}_1) = -1/r$. Hence

$$\omega_{13}(E_1) = 0$$

whilst

$$\bar{\omega}_{13}(x_*(E_1)) = \bar{\omega}_{13}(\bar{E}_1) = -1/r.$$

Thus x does not preserve ω_{13}. It is true that in this case x preserves ω_{23}, but as the labelling of E_1, E_2 is arbitrary it is clear that ω_{23} is not preserved in general. Contrast this with Lemma VI.5.3(2), which shows that the 1-form ω_{12} *is* an isometric invariant.

We can use the fact that k_1 and k_2 are not in general preserved by isometries to show that the shape operator S is also not preserved in general. The shape operator would be preserved by an isometry F if

$$\bar{S}(F_*(v)) = F_*(S(v))$$

for all tangent vectors v tangent to the domain surface; that is, if the linear transformations S and $F_*^{-1}\ \bar{S}\ F_*$ were equal at each point p of the domain surface. Now the eigenvalues of S are k_1 and k_2; the eigenvalues of $F_*^{-1}\ \bar{S}\ F_*$ are equal to the eigenvalues of $\bar{S}$, which are $\bar{k}_1$ and $\bar{k}_2$: in general these are not the same as k_1 and k_2 and so S and $F_*^{-1}\ \bar{S}\ F_*$ cannot be equal. Thus the shape operator is not preserved.

Tangent vectors v tangent to E^2 with some special property are said to be "preserved by x" if their images $x_*(v)$ have the same property. We note that x_* is onto because x is regular (see the second paragraph on page 161). E^2 is all-umbilic and so every unit tangent vector v tangent to E^2 is a principal vector. Their images $x_*(v)$ are all the unit tangent vectors tangent to M (because x_* is onto and x is a local isometry) and not all of these are principal vectors as there are only four principal vectors at each point of M. Thus x does not preserve principal vectors.

Curves are treated similarly. Curves α in E^2 with a special property are "preserved by x" if their images $x(\alpha)$ have the same property. All curves α in E^2 are principal curves : since x is onto, Lemma IV.3.1, shows that their images $x(\alpha)$ are all of the curves in M, not all of which are principal, as the principal curves in M are just the rulings and the cross-sectional circles, so x does not preserve principal curves. Similarly x does not preserve asymptotic curves because all curves in E^2 are asymptotic but only the rulings on M are.

3. Page 275, Exercise 3.

Let I_M, I_N be the curvature intervals of M, N respectively. We show equality by showing

$$I_M \subset I_N \text{ and } I_N \subset I_M.$$

We have to show both inclusions, because a local isometry which is not actually an isometry does not have an inverse. Suppose $x \in I_M$. Then there is some point p in M with $K(p) = x$. Since local isometries preserve Gaussian curvature

$$K(p) = \bar{K}(F(p)).$$

Thus $x = \bar{K}(F(p))$. Now $F(p) \in N$ so $x \in I_N$. Arguing thus for each $x \in I_M$, we have $I_M \subset I_N$.

Now suppose $y \in I_N$. Then $y = \bar{K}(q)$ for some point q in N. F is onto, so $q = F(p)$ for some p in M. As above

$$K(p) = \bar{K}(F(p)) = \bar{K}(q) = y$$

and so $y \in I_M$. Thus $I_N \subset I_M$ and so $I_M = I_N$.

There are many examples which show that the converse is false. Here is one : the saddle surface M : $z = xy$ has curvature interval $[-1, 0)$ (see page 215) and the helicoid covered by

$$x(u, v) = (u \cos v, u \sin v, bv)$$

has curvature interval $[-1/b^2, 0)$ (see page 214). If we take the helicoid with $b = 1$ then these two surfaces have the same curvature interval but there is no local isometry from one onto the other. The helicoid has Gaussian curvature -1 at all points with $u = 0$, but the saddle surface has Gaussian curvature -1

only at the origin. Thus if F were a local isometry from the helicoid to the saddle surface it would have to map the entire z axis to the origin. Suppose p is a point on the z axis. Then there would be a small neighbourhood of p in the helicoid on which F was an isometry. Such a neighbourhood would have to include other points q on the z axis. Since p and q would both be mapped to $\mathbf{0}$ by F the intrinsic distance between them would not be preserved, contradicting Theorem VI.4.3.

4. Page 275, Exercise 2.

Example VI.4.6(1) tells us that the patch x from E^2 to the circular cylinder $M : x^2 + y^2 = r^2$ given by

$$x : (u, v) \longmapsto (r\cos(u/r), r\sin(u/r), v)$$

is a local isometry. Hence, by the result of Exercise 1, we know that if α is a geodesic in E^2 then $x(\alpha)$ is a geodesic in M. Moreover, because the restriction of x to small enough regions is an isometry, all geodesics on M are of the form $x(\alpha)$ for some geodesic α on E^2. The non-constant geodesics in E^2 are simply the straight lines (see Example V.5.8(1) on page 228)

$$t \longmapsto p + tq$$

where $p, q \in E^2$, $q \neq \mathbf{0}$. If α is the geodesic

$$\alpha : t \longmapsto (p_1, p_2) + t(q_1, q_2)$$

then

$$x(\alpha) : t \longmapsto x(p_1 + tq_1, p_2 + tq_2)$$
$$= \left(r\cos\left(\frac{p_1 + tq_1}{r}\right), r\sin\left(\frac{p_1 + tq_1}{r}\right), p_2 + tq_2\right).$$

Putting $a = q_1/r$, $b = p_1/r$, $c = q_2$, $d = p_2$ shows that the non-constant geodesics on M are the curves of the form

$$t \longmapsto (r\cos(at + b), r\sin(at + b), ct + d),$$

as found in Example V.5.8(3) on page 229.

5. Page 275, Exercise 4.

We have the following curvature intervals:

sphere	$\{1/r^2\}$	
torus	$[-1/r(R-r), 1/r(R+r)]$	(see page 236)
helicoid	$[-1/b^2, 0)$	(see page 214)
circular cylinder	$\{0\}$	
saddle surface	$[-1, 0)$	(see page 215)

There are no values of r, R, b for which any two of these intervals can be equal except those of the helicoid ($b = 1$) and the saddle surface, which we have already shown (Exercise 3) are not isometric. By the main result of Exercise 3, no two of these surfaces can be isometric.

VI.6 ORTHOGONAL COORDINATES

Introduction

This section follows on from Section VI.1 and VI.2 and develops a simple formula for the Gaussian curvature of a region described by an orthogonal patch.

READ: Section VI.6 (pages 276-279).

Supplementary Comment

(i) *Page 277 : Equation (3)* Since θ_1, θ_2 form a "basis" for the 1-forms so do du and dv. Hence $\omega_{12} = a\,du + b\,dv$. The first structural equations become

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$= a\,du \wedge \theta_2, \qquad \text{since } \theta_2, = \sqrt{G}\,dv,$$

and

$$d\theta_2 = \omega_{21} \wedge \theta_1 = -\omega_{12} \wedge \theta_1$$

$$= -b\,dv \wedge \theta_1.$$

Comparison with the equations on lines −13 and −14 gives

$$a = \frac{-(\sqrt{E})_v}{\sqrt{G}}, \quad b = \frac{(\sqrt{G})_u}{\sqrt{E}} \qquad \text{and the formula for } \omega_{12} \text{ follows.}$$

Summary

Definitions

(i)	Orthogonal patch	Page 276, line −14
(ii)	Associated frame field	Page 276, Definition 6.1

Results

(i)	$E_1 = x_u/\sqrt{E}$ and $E_2 = x_v/\sqrt{G}$.	Page 277, Equation (1)
(ii)	$\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$.	Page 277, Equation (2)
(iii)	$\omega_{12} = -\frac{(\sqrt{E})_v}{\sqrt{G}}\,du + \frac{(\sqrt{G})_u}{\sqrt{E}}\,dv.$	Page 277, Equation (3)
(iv)	$K = -\frac{1}{\sqrt{(EG)}}\left\{\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u + \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right\}$	Page 278, Lemma 6.3

Technique

(i) Computing θ_1, θ_2, ω_{12}, K for an orthogonal patch by the technique that was used to obtain the above results.

Exercise

Technique (i)

1. Page 279, Exercise 1(a).

Solution

1. Page 279, Exercise 1(a).

$x_u = (\cos v, \sin v, 0)$ $\qquad E = 1$

$x_v = (-u \sin v, u \cos v, b)$ $\qquad F = 0$

$G = u^2 + b^2$

$$\theta_1 = \sqrt{E}\, du = du, \quad \theta_2 = \sqrt{G}\, dv = (u^2 + b^2)^{\frac{1}{2}} dv$$

$$d\theta_1 = 0, \quad d\theta_2 = \frac{u}{(u^2 + b^2)^{\frac{1}{2}}} du\, dv = \frac{-u dv}{(u^2 + b^2)^{\frac{1}{2}}} \wedge \theta_1$$

$$\omega_{12} = \frac{-(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv = \frac{u dv}{(u^2 + b^2)^{\frac{1}{2}}}$$

$$d\omega_{12} = \frac{b^2}{(u^2 + b^2)^{\frac{3}{2}}} du\, dv = \frac{b^2}{(u^2 + b^2)^2} \theta_1 \wedge \theta_2$$

Hence $K = \dfrac{-b^2}{(u^2 + b^2)^2}$.

VI.8 CONGRUENCE OF SURFACES

Introduction

This section draws on the results of all the previous sections of O'Neill in M334 except Section VI.3.

At last we relate the concept of "isometry of E^3" (Chapter III) with that of "isometry of surfaces" (Sections VI.4 and VI.5). We find that Euclidean isometries do induce isometries of surfaces, but that the converse is not true : isometries of surfaces need to preserve shape operators in order to be realizable as Euclidean isometries. In fact the shape operator (Chapter V) turns out to be the analogue for surfaces of the curvature and torsion functions (Section II.3) for curves, in that it determines the congruence of surfaces precisely.

You should be familiar with Section III.5 before reading this section, as the ideas and results developed here are similar to those (but a little harder!). In particular some details from the proof of Theorem III.5.7 are needed in the proof of Theorem VI.8.3 : you should read this latter proof, but we do not expect you to be able to reproduce it.

READ: Section VI.8 (pages 297-301).

Supplementary Comments

(i) *Page 297 : line −7* The fact that the restriction $F : M \longrightarrow \bar{M}$ is a mapping is a consequence of Comment (i) on page 33 of Text, Part IV.

(ii) *Page 298 : last four lines of proof*

$$\begin{aligned}
F_*(S(v)) &= F_*(-\nabla_v U), && \text{because } S(v) \text{ is defined to be } -\nabla_v U,\\
&= F_*(-U'(0)), && \text{(where } U \text{ is restricted to } \alpha\text{) by the result of Exercise V.1.1,}\\
&= -F_*(U'(0)), && \text{because } F_* \text{ is linear,}\\
&= -(F_*(U))'(0), && \text{because } F_* \text{ preserves derivatives of vector fields (Corollary III.4.1),}\\
&= -\bar{U}'(0), && \text{because } \bar{U} = F_*(U).
\end{aligned}$$

Restricting U to α automatically restricts $\bar{U}$ to $F(\alpha)$, which is a curve in $\bar{M}$ with initial velocity $F_*(v)$: hence the result of Exercise V.1.1 can be used again to give

$$\bar{S}(F_*(v)) = -\bar{U}'(0)$$

and so

$$\bar{S}(F_*(v)) = F_*(S(v)).$$

(iii) *Page 299 : proof of Lemma 8.2* The precise form of Corollary VI.1.5 that is used is

$$S(v)\cdot E_i(p) = \omega_{i3}(v)$$

and the similar expression for $\bar{M}$.

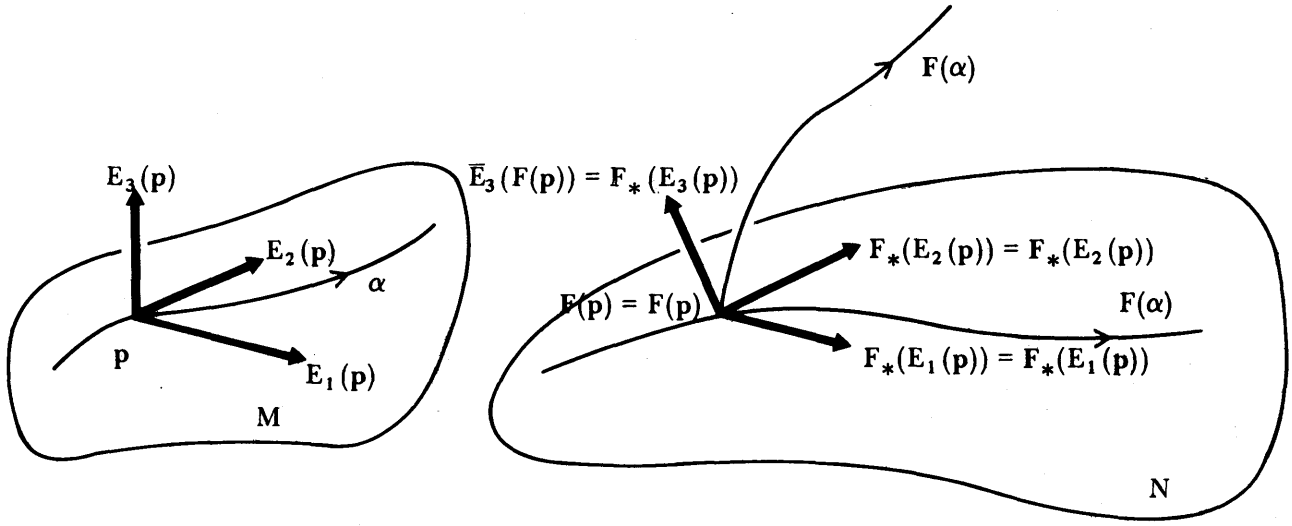
(iv) *Page 299 : lines −7 to −2* Let $E_1(p)$, $E_2(p)$ be a tangent frame to M at p. Then $E_1(p)$, $E_2(p)$, $E_3(p)$ is a frame at p. Moreover $F_*(E_1(p))$, $F_*(E_2(p))$ is a tangent frame to $\overline{M}$ at F(p), and so $F_*(E_1(p))$, $F_*(E_2(p))$, $\overline{E}_3(F(p))$ is a frame at F(p). By Theorem III.2.3, there is a Euclidean isometry F such that

$$F(p) = F(p)$$
$$F_*(E_1(p)) = F_*(E_1(p))$$
$$F_*(E_2(p)) = F_*(E_2(p))$$
$$F_*(E_3(p)) = \overline{E}_3(F(p)).$$

Now if $v \in T_p(M)$, $v = aE_1(p) + bE_2(p)$ for some real numbers a and b, so

$$\begin{aligned} F_*(v) &= F_*(aE_1(p) + bE_2(p)) \\ &= aF_*(E_1(p)) + bF_*(E_2(p)) \\ &= aF_*(E_1(p)) + bF_*(E_1(p)) \\ &= F_*(aE_1(p) + bE_2(p)) \\ &= F_*(v). \end{aligned}$$

(v) *Page 300 : lines 1, 2* We have F and F agreeing at p, and want to show that F(q) = F(q) for all q in M. For a particular q we choose a curve α in M from p to q. Then $F(\alpha)$ is a curve in N and $F(\alpha)$ is a curve in E^3.


$E_3(p)$
$E_2(p)$
α
p
$E_1(p)$
M
$F(\alpha)$
$\overline{E}_3(F(p)) = F_*(E_3(p))$
$F_*(E_2(p)) = F_*(E_2(p))$
F(p) = F(p)
$F(\alpha)$
$F_*(E_1(p)) = F_*(E_1(p))$
N

It suffices to show that $F(\alpha) = F(\alpha)$, for then F(q) = F(q) because $q = \alpha(t)$ for some t.

(vi) *Page 301 : last four lines of proof* We compare the equations (C1) and (C2):

$$(F_*E_i)' \cdot F_*E_j = \overline{E}_i' \cdot \overline{E}_j$$
$$F(\alpha)' \cdot F_*E_i = \bar{\alpha}' \cdot \overline{E}_i$$

where $\{F_*E_i\}$, $\{\overline{E}_i\}$ are frame fields on $F(\alpha)$, $\bar{\alpha}$ respectively, with the conditions for Theorem III.5.7:

$$E_i' \cdot E_j = F_i' \cdot F_j$$
$$\alpha' \cdot E_i = \beta' \cdot F_i$$

where $\{E_i\}$, $\{F_i\}$ are frame fields on α, β respectively. We see that $F(\alpha)$, $\bar{\alpha}$, $\{F_* E_i\}$, $\{\bar{E}_i\}$ satisfy all the conditions for Theorem III.5.7, and so we can draw the conclusion of that theorem, namely that $F(\alpha)$ and $\bar{\alpha}$ are congruent. We may assume that $\alpha(0) = p$. Then the details at the beginning of the proof of Theorem III.5.7 show that the Euclidean isometry G carrying $F(\alpha)$ to $\bar{\alpha}$ is the unique Euclidean isometry which takes $F_* E_i$ at $F(\alpha(0))$ to $\bar{E}_i$ at $\bar{\alpha}(0)$ for $i = 1, 2, 3$. Now $\alpha(0) = p$, so

$$F(\alpha(0)) = F(p)$$

and

$$\bar{\alpha}(0) = F(\alpha(0)) = F(p).$$

Moreover, for $i = 1, 2, 3$, $(F_*(E_i))(F(p)) = F_*(E_i(p))$;

for $i = 1, 2$, $F_*(E_i(p)) = F_*(E_i(p))$, by the choice of F on page 299,

$= \bar{E}_i(F(p))$, by the choice of $\bar{E}_i$ on page 300,

and $F_*(E_3(p)) = \bar{E}_3(F(p))$, by the choice of F on page 299.

Thus G leaves invariant the point $F(p)$, and the frame $F_*(E_i(p))$ at that point, and so G, which is unique, must be the identity. Hence

$$F(\alpha) = \bar{\alpha} = G(F(\alpha)) = F(\alpha).$$

Summary

Definition

(i)	Congruent surfaces	Page 297, line 1 of the section

Example

(i)	The saddle surfaces $M: z = xy$ and $M: z = (x^2 - y^2)/2$ are congruent.	Page 297, line 5 of the section

Results

(i)	The restriction of a Euclidean isometry to an oriented surface in E^3 is an isometry of surfaces which preserves shape operators, up to the usual ambiguity in sign (congruent surfaces are isometric).	Page 297, Theorem 8.1
(ii)	If F is an isometry of oriented surfaces which preserves the shape operators defined in terms of the orienting unit normals E_3 and $\bar{E}_3$ and which takes the tangent frame E_1, E_2 to the tangent frame $\bar{E}_1, \bar{E}_2$, then $F^*(\bar{\omega}_{ij}) = \omega_{ij}, (1 \leqslant i, j \leqslant 3)$.	Page 299, Lemma 8.2
(iii)	If F is an isometry of oriented surfaces that preserves shape operators then F is the restriction to the surface concerned of a Euclidean isometry.	Page 299, Theorem 8.3
(iv)	Euclidean isometries preserve mean, Gaussian and principal curvatures, principal vectors, umbilics, asymptotic and principal curves and geodesics.	Text, Page 41, Exercise 1

Techniques

(i)	Finding properties of surfaces invariant under Euclidean isometries.	Page 297, Theorem 8.1 and Page 299, Lemma 8.2
(ii)	Finding Euclidean isometries between surfaces.	Page 299, proof of Theorem 8.3
(iii)	Using Result (iii).	Page 299, Theorem 8.3
(iv)	Using results about congruence of curves to obtain results about congruence of connected surfaces.	

Exercises

Technique (i)

1. Page 301, Exercise 3. (HINT : Choose a unit normal (and hence shape operator) on each of M, N and use Theorem VI.8.1.)

Technique (ii)

2. Page 303, Exercise 9. (HINT : First show that $F(0) = 0$ in each case, using Theorem VI.8.1 and the theorema egregium. Then use Result (iv) and Exercise V.2.1(b)). A Euclidean symmetry is defined in Exercise 7.

Technique (iii)

3. Page 302, Exercise 4.

Technique (iv)

4. Page 301, Exercise 2. The tangent surface is defined in Exercise V.5.11(b). (HINT : Use Theorem III.5.3 to show that there is a Euclidean isometry F with $F(\alpha) = \beta$; write $F = T_aC$ in the usual way and show that

$$F(\alpha(u) + vT(u)) = \beta(u) + v\bar{T}(u),$$

where $T, \bar{T}$ are the unit tangent vector fields on α, β respectively.)

Solutions

1. Page 301, Exercise 3.

Since $F|M$ is, by Theorem VI.8.1, an isometry of surfaces, $F|M$ preserves Gaussian curvature and geodesics as these are preserved by all isometrics of surfaces.

Writing F for $F|M$, we know that F preserves shape operators, that is

$$F_*(S(v)) = \bar{S}(F_*(v))$$

for all v tangent to M. If $S(v) = kv$ then

$$\bar{S}(F_*(v)) = F_*(S(v)) = F_*(kv) = kF_*(v)$$

and so F takes eigenvectors and eigenvalues of S into eigenvectors and eigenvalues of $\bar{S}$; that is, F preserves principal vectors (and hence principal directions) and principal curvatures. The mean curvature is the arithmetic mean of the principal curvatures, and so this is also preserved. Moreover, if p is umbilic then all tangent vectors in $T_p(M)$ are eigenvectors so all tangent vectors in $T_{F(p)}(N)$ are eigenvectors, so F(p) is also umbilic.

Now let α be a curve in M. If α is asymptotic then $S(\alpha') = 0$ so

$$\overline{S}(F(\alpha)') = \overline{S}(F_*(\alpha')) = F_*(S(\alpha')) = F_*(0) = 0$$

and so $F(\alpha)$ is asymptotic also. If α is principal then α' is always an eigenvector of S so $F(\alpha)' = F_*(\alpha')$ is always an eigenvector of $\overline{S}$, so $F(\alpha)$ is principal also.

We saw in Section VI.5 that only Gaussian curvature and geodesics are preserved by arbitrary isometries.

2. Page 303, Exercise 9.

Suppose F is a Euclidean symmetry of M; that is, F is a Euclidean isometry and F(M) = M. Then by Theorem VI.8.1 $F|M$ is an isometry of surfaces, and so $F|M$ preserves Gaussian curvature by the theorema egregium.

M : z = xy, and so every point p on M has the form (p_1, p_2, p_1p_2). We have seen before (page 215) that the Gaussian curvature at the point $(p_1, p_2, p_1 p_2)$ of M is $-(1 + p_1^2 + p_2^2)^{-2}$, which is -1 only at 0, and so F must leave the point 0 invariant. Now by Lemma III.1.6 F is an orthogonal transformation.

Clearly F cannot be just any orthogonal transformation. Result (iv) shows that F must preserve the principal vectors at 0. We saw in Exercise V.2.1 that these are

$$\pm(\mathbf{u}_1 \pm \mathbf{u}_2)/\sqrt{2}$$

where $\mathbf{u}_1 = U_1(0)$, $\mathbf{u}_2 = U_2(0)$. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ make angles of $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ with the principal vectors. The derivative map F_* preserves dot products (and hence angles) so $F_*(\mathbf{u}_1)$ and $F_*(\mathbf{u}_2)$ are orthogonal unit vectors making angles of $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ with the principal vectors. This gives the eight possibilities:

$$F_*(\mathbf{u}_1) = \pm\,\mathbf{u}_i,\ F_*(\mathbf{u}_2) = \pm\,\mathbf{u}_j,\ (\{i, j\} = \{1, 2\}).$$

Since F is an orthogonal transformation, F and F_* have the same matrix C representing them (Theorem III.2.1). We have shown that the first two columns of C must be one of:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}$$

and since C is orthogonal this means that the third column must be

$$\begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix} \quad \text{where } \varepsilon = \pm 1.$$

We test the 16 possibilities we now have by using the fact that F(M) = M only if

$$C\begin{pmatrix} x \\ y \\ xy \end{pmatrix} \quad \text{is of the form} \quad \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

Since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} x \\ y \\ \varepsilon xy \end{pmatrix}$$

we can have $\varepsilon = 1$ but not $\varepsilon = -1$ in this case;
Since

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} -x \\ y \\ \varepsilon xy \end{pmatrix}$$

we can have $\varepsilon = -1$ but not $\varepsilon = 1$ in this case : working similarly we find that the eight Euclidean symmetries of M are the orthogonal transformations with matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

F carries M *onto* M in each case because the point (x, y, xy) is the image of one of the eight points (±x, ±y, (±x)(±y)), (±y, ±x, (±y)(±x)).

3. Page 302, Exercise 4.

Suppose Σ has radius r, Σ' has radius R. If $p \in \Sigma$, $K(p) = 1/r^2$ and $\bar{K}(F(p)) = 1/R^2$. Since F is an isometry of surfaces F preserves Gaussian curvature and so r = R.

Orient Σ and Σ' by their outward normals. Then (see page 192) $S(v) = -v/r$ for tangent vectors v tangent to Σ and $\bar{S}(w) = -w/r$ for tangent vectors w tangent to Σ'. In particular if we take $w = F_*(v)$ we obtain

$$F_*(S(v)) = F_*(-v/r) = -F_*(v)/r = \bar{S}(F_*(v)).$$

Thus F is an isometry of surfaces preserving shape operators, and so by Theorem VI.8.3 there is a Euclidean isometry F such that $F = F|M$.

4. Page 301, Exercise 2.

We have the surfaces M_α and M_β parametrized by

$$x : (u, v) \longmapsto \alpha(u) + vT(u)$$

$$y : (u, v) \longmapsto \beta(u) + v\bar{T}(u)$$

respectively, where

$$\alpha'(u) = T(u)_{\alpha(u)}$$

$$\beta'(u) = \bar{T}(u)_{\beta(u)}.$$

Since $\kappa_\alpha = \kappa_\beta > 0$ and $\tau_\alpha = \tau_\beta$, Theorem III.5.3 shows that there is a Euclidean isometry F such that $F(\alpha) = \beta$. We aim to show that $F(M_\alpha) = M_\beta$ to demonstrate that M_α is congruent to M_β, and we can do this by showing

$$F(\alpha(u) + vT(u)) = \beta(u) + v\bar{T}(u).$$

Write $F = T_aC$, where T_a is translation by a and C is an orthogonal transformation, as in Chapter III. Applying Theorem III.2.1 to $\alpha'(u)$ gives

$$F_*(\alpha'(u)) = F_*(T(u)_{\alpha(u)}) = CT(u)_{F(\alpha(u))}.$$

However, $F(\alpha) = \beta$ and $F_*(\alpha') = (F(\alpha))' = \beta'$, so

$$CT(u)_{\beta(u)} = \beta'(u) = \bar{T}(u)_{\beta(u)}$$

and so

$$CT(u) = \bar{T}(u).$$

Now

$$\begin{aligned} F(\alpha(u) + vT(u)) &= C(\alpha(u) + vT(u)) + a \\ &= C\alpha(u) + vCT(u) + a, \qquad \text{because C is linear,} \\ &= (C\alpha(u) + a) + vCT(u) \\ &= F(\alpha(u)) + vCT(u) \\ &= \beta(u) + v\bar{T}(u). \end{aligned}$$

SUMMARY AND LOOK-AHEAD

You have now completed all those sections of O'Neill that are a part of M334. O'Neill gives a short summary on page 303, which you may like to read, although the best summary of this course is probably given by the structure diagrams of Part 0, which should now make more sense to you.

There are, however, still one Chapter and several isolated sections of O'Neill which you have so far not had to read but may someday want to.

To begin with there are Sections IV.6 and VI.7, which deal with integration of forms on a surface: this ties in with the work on Green's theorem in M321.

In Section IV.8 O'Neill tries to give a more general approach to the theory of surfaces which underlies the material of Chapter VII on Riemannian geometry. The approach of Chapter VII is typical of generalizations – theorems of previous sections now become definitions.

In Section VII.1 Riemannian surfaces are introduced as surfaces that are equipped with an arbitrary dot product on their tangent spaces, though in all the examples dealt with the tangent spaces and dot products are derived directly from the usual ones. O'Neill assumes that vector fields, 1-forms, 2-forms all exist on such surfaces and that all the results of the exterior calculus still hold. For instance, he assumes the existence of frame fields and dual forms and then uses the analogue of the structure equations $d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\omega_2 = \omega_{21} \wedge \theta_1$ to define the connection form ω_{12}.

In Section VII.2 the Gaussian curvature is defined using the second structural equation

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

and it is calculated for several important examples such as the stereographic plane and the hyperbolic plane.

Working backwards in Section VII.3 he shows that it is possible to construct a unique covariant derivative satisfying the connection equations

$$\nabla_V E_1 = \omega_{12}(V)E_2, \qquad \nabla_V E_2 = \omega_{21}(V)E_1,$$

though when we are dealing with a surface in E^3 this turns out to be the restriction of the usual covariant derivative to the tangent planes in question.

This covariant derivative is then used to work backwards to a definition of derivative of a vector field along a curve, and in Section VII.4 this is used to investigate geodesics on a Riemannian surface.

In Sections VII.5 and VII.6 the geodesics are shown to minimize, at least locally, the distance between points, and it is shown how measurement of geodesic distance can be used by "inhabitants" in a surface to determine its Gaussian curvature.

In the final sections of Chapter VII a relationship is found between the Gaussian curvature of a surface, a local property, and the Euler-Poincaré characteristic, a global property, which provides a link between differential geometry and the further disciplines of differential and algebraic topology.

FURTHER EXERCISES AND SOLUTIONS

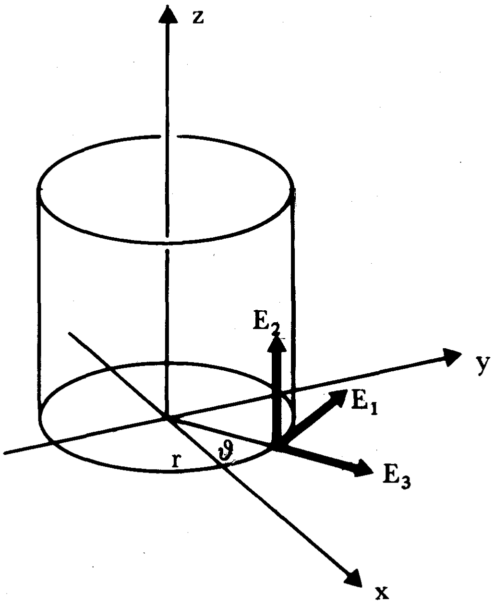
Section VI.1

Technique Exercise

Page 251, Exercise 5.

Solution

5.



$E_1 = -\sin\vartheta U_1 + \cos\vartheta U_2$, $E_2 = U_3$, $\quad E_3 = \cos\vartheta U_1 + \sin\vartheta U_2$

$\theta_1 = r\,d\vartheta$, $\quad \theta_2 = dz$ by page 96, Exercise 4.

$$\omega = \begin{pmatrix} 0 & 0 & -d\vartheta \\ 0 & 0 & 0 \\ d\vartheta & 0 & 0 \end{pmatrix}$$

by page 90, line 2 and a suitable permutation.

$$S(v) = -d\vartheta(v)E_1 = -\frac{1}{r}\theta_1(v)E_1$$

$$S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \end{pmatrix}$$

Section VI.2

Technique Exercise

Page 255, Exercise 1.

Solution

1. $$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2) = 0$$

$$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1) = \frac{R\sin\varphi}{r(R + r\cos\varphi)^2}$$

Section VI.3

No further exercises are recommended.

Section VI.4

Technique Exercise

Page 269, Exercise 2.

Other Recommended Exercises

Page 269, Exercise 4. This generalizes Theorem VI.4.3.

Page 270, Exercise 7 (second part). This generalizes Lemma VI.4.5.

Page 270, Exercise 10. This also generalizes Lemma VI.4.5.

Solutions

2(a). $\|F(\alpha')\| = \|\alpha'\|$ for all curves α in M

$\Leftrightarrow \|F_*v\| = \|v\|$ for all vectors v tangent to M.

2(b). $\|F(\alpha')\| = \|\alpha'\|$

$$\Rightarrow L(F(\alpha)) = \int_a^b \|F(\alpha)'\| = \int_a^b \|\alpha'\| = L(\alpha).$$

F preserves the lengths of all sub-curves of α

$$\Rightarrow \int_c^d \|F(\alpha')\| = \int_c^d \|\alpha'\| \quad \text{for all } c, d$$

$$\Rightarrow \|F(\alpha')\| = \|\alpha'\|.$$

4. If $F : M \longrightarrow \overline{M}$ is a local isometry and $p, q \in M$ then

$$\rho(p, q) \geqslant \rho(F(p), F(q)).$$

Proof:
$$\begin{aligned}\rho(p, q) &= \text{g.l.b. } \{L(\alpha) : \alpha \text{ is a curve segment from } p \text{ to } q \text{ in } M\}\\ &= \text{g.l.b. } \{L(F(\alpha)) : \alpha \text{ is a curve segment from } p \text{ to } q \text{ in } M\}\\ &\geqslant \text{g.l.b. } \{L(\beta) : \beta \text{ is a curve segment from } F(p) \text{ to } F(q) \text{ in } \overline{M}\}\end{aligned}$$

because while every $F(\alpha)$ is a curve segment from $F(p)$ to $F(q)$ in $\overline{M}$, there may be curve segments β from $F(p)$ to $F(q)$ in $\overline{M}$ which are not of the form $F(\alpha)$.

7. $E = (\lambda(u, v))^2 = G$

$F = (\lambda(u, v))^2 \cdot 0 = 0$

10(a). They are mapped to (some of) the parameter curves of $\bar{x}$.

10(b). $x_u \cdot x_u = E$; $\quad \bar{x}_u \cdot \bar{x}_u = (\frac{\partial f}{\partial u}\bar{x}_u(f, g)) \cdot (\frac{\partial f}{\partial u}\bar{x}_u(f, g))$

$$= (\frac{\partial f}{\partial u})^2 \bar{E}(f, g).$$

10(c). Multiply all the right-hand sides by λ^2, where λ is a function of u, v.

Section VI.5

Technique Exercise

Page 276, Exercise 6. (Ignore the comment in brackets.) Assume the helicoid has $b = 1$.

Solution

6. Parametrize H by

$$x(u, v) = (u\cos v, u\sin v, v)$$

and C by

$$y(s, \vartheta) = (\sinh^{-1} s, (1 + s^2)^{\frac{1}{2}} \cos\vartheta, (1 + s^2)^{\frac{1}{2}} \sin\vartheta).$$

On H, $K(x(u, v)) = -(1 + u^2)^{-2} = -1$ on and only on the axis $u = 0$;

on C, $K(y(s,\vartheta)) = -(1 + s^2)^{-2} = -1$ on and only on the minimal circle $s = 0$.

The ruling $v = v_0$ on H is a unit-speed curve α with Gaussian curvature at $\alpha(u)$ given by $K(\alpha(u)) = -(1+u^2)^{-2}$. Put $F(\alpha) = y(s(u), \vartheta(u))$. Then $K(F(\alpha(u)) = -(1+u^2)^{-2} \Rightarrow s(u) = \pm u$ and so $\|F(\alpha)'\| = 1 \Rightarrow \vartheta'(u) = 0 \Rightarrow \vartheta(u)$ is constant $\Rightarrow F(\alpha)$ is a meridan of C.

Section VI.6

Technique Exercises

Pages 279-280, Exercises 1(b), 1(c), 3,4.

Solutions

1(b). $\theta_1 = (1 + u^2)^{\frac{1}{2}} du \qquad \theta_2 = u dv$

$\omega_{12} = dv/(1 + u^2)^{\frac{1}{2}}$

$K = 1/(1 + u^2)^2$

1(c). $E_1 = x_u/(1 + a^2)^{\frac{1}{2}} \qquad E_2 = x_v/u$

$\theta_1 = (1 + a^2)^{\frac{1}{2}} du \qquad \theta_2 = u dv$

$\omega_{12} = dv/(1 + a^2)^{\frac{1}{2}} \qquad K = 0$

3. $E_1 = x_u \qquad E_2 = (x_v - \cos\vartheta\, x_u)/\sin\vartheta$

$\theta_1 = du + \cos\vartheta\, dv \qquad \theta_2 = \sin\vartheta\, dv$

$\omega_{12} = -\vartheta_u\, du$

$K = -\vartheta_{uv}/\sin\vartheta$

4(a). $S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$

$$\frac{\ell}{E}x_u = S(x_u) = \omega_{13}(x_u)\frac{x_u}{\sqrt{E}} + \omega_{23}(x_u)\frac{x_v}{\sqrt{G}}$$

$$\frac{n}{G}x_v = S(x_v) = \omega_{13}(x_v)\frac{x_u}{\sqrt{E}} + \omega_{23}(x_v)\frac{x_v}{\sqrt{G}}$$

$$\omega_{13}(x_u) = \frac{\ell}{\sqrt{E}} \qquad \omega_{13}(x_v) = 0 \qquad \omega_{13} = \frac{\ell}{\sqrt{E}}\,du$$

$$\omega_{23}(x_u) = 0 \qquad \omega_{23}(x_v) = \frac{n}{\sqrt{G}} \qquad \omega_{23} = \frac{n}{\sqrt{G}}\,dv$$

4(b). Use the Codazzi equation

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}.$$

$$d\omega_{13} = \left(\frac{\ell E_v}{2E} - \ell_v\right)\frac{dudv}{\sqrt{E}}$$

$$\omega_{12} \wedge \omega_{23} = \frac{-n(\sqrt{E})_v}{G}dudv = -\frac{nE_v}{2G}\;\frac{dudv}{\sqrt{E}}$$

Hence $\ell_v = \frac{1}{2}(\frac{\ell}{E} + \frac{\ell}{G})\,E_v = HE_v$, by Exercise V.4.9.

Using $d\omega_{23} = \omega_{21} \wedge \omega_{13}$ we find similarly that $n_u = HG_u$.

Section VI.8

Technique Exercises

Pages 302-3, Exercises 5, 10.

Other Recommended Exercises

Page 302, Exercises 7, 8. These relate Euclidean isometries of a surface to group theory.

Solutions

5(a). $C(p_1, p_2, p_1p_2) = (-p_1, p_2, -p_1p_2)$.

5(b). M, as the image of F, must be oriented by $-U$.

10. K is maximal at $(\pm\alpha, 0, 0)$ and minimal at $(0, 0, \pm c)$. The Euclidean symmetries are the eight orthogonal transformations

$$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$$

7(a). Closure and the inclusion of inverses and the identity are trivial to prove.

7(b). If $F : M \longrightarrow N$ is a Euclidean isometry then

$$G \longmapsto F^{-1}GF$$

is an isomorphism of $S(M)$ onto $S(N)$.

8. By Exercise 7(b) we may assume Σ has centre the origin. Let $F = TC$.

$$\begin{aligned} F(\Sigma) = \Sigma &\Leftrightarrow F(0) = 0 \\ &\Leftrightarrow T = I \\ &\Leftrightarrow F = C. \end{aligned}$$

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3